A SUCCESSIVE APPROXIMATION METHOD FOR SOLVING A NON-LINEAR VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND*

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A NUMBER of problems of heat transmission with non-linear heat exchange conditions at the phase separation boundary [1, 2], and also mass transfer processes accompanied by surface transformations of the transferred matter (for example, as the result of adsorption, surface reaction, an electrodynamic transformation etc.), lead to a non-linear Volterra integral equation of the second kind with an integrable singularity in the kernel

$$y(x)=\varphi(x)+\int_0^x \frac{\bar{k}(x,t)}{(x-t)^\alpha} f[y(t)]\,dt, \qquad 0<\alpha<1, \tag{1}$$

where $f(y)$ is given non-linear function, $\bar{k}(x, t)\big|_{t=x} \neq 0$.

A standard method of successive approximations can be used to construct the solution of Eq. (1). However, for a fairly wide range of x a good representation of the solution of (1) can be attained only for a large number of terms of the series of successive approximations. Of course, the practical use of such a solution is extremely difficult. Hence, many attempts have been made [1, 3] to perfect a standard iteration method, in order that the first terms of the series alone may more satisfactorily describe the solution of Eq. (1) over a sufficiently wide range of x.

In [4] one of the methods for the approximate solution of (1) for the particular case $\varphi(x)$, $k(x, t)$ and $\alpha = ½$ was proposed. The method was based on the properties of the kernel of the integral equation (1) and on the assumption of the comparatively slow variation of $y(x)$. The approximate solution for the particular case of Eq. (1) found in [4] was compared with the solution obtained on a computer. It appeared that their difference does not exceed 8% for all $x \geqslant 0$.

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Below we describe a method of successive approximations which is an efficient method in a number of cases [3]. The proposed method is basically similar to the idea used in [5] for the regularisation of Fredholm equations.

In the following discussion we will have in mind primarily problems on mass transfer accompanied by the adsorption of the transferred matter at the phase separation boundary. If the adsorption process is considered to be sufficiently rapid, this stage can be described by means of adsorption isotherms. Then the form of the function $f(y)$ in Eq. (1) is determined by the form of the adsorption isotherms and satisfies the following conditions:

$$f(y) \text{ is defined and continuous for } -\infty < y < b,\ b > 0,$$
$$\lim_{y \to b-0} f(y) = +\infty. \tag{2}$$

We will consider only non-negative values of the argument, that is, $x \geqslant 0$. We put formally $y_n \equiv y(x)$ and rewrite (1) in the form

$$y_n = \varphi(x) + \int_0^x \frac{\bar{k}(x,t)}{(x-t)^\alpha} f(y_n)\, dt$$

or

$$y_n = \varphi(x) + \int_0^x \frac{\bar{k}(x,t)}{(x-t)^\alpha} f(y_{n-1})\, dt + \int_0^x \frac{\bar{k}(x,t)}{(x-t)^\alpha} [f(y_n) - f(y_{n-1})]\, dt. \tag{3}$$

We note that in the last term on the right side of (3) the denominator of the kernel vanishes for $t = x$. Hence we put approximately

$$\int_0^x \frac{\bar{k}(x,t)}{(x-t)^\alpha} [f(y_n) - f(y_{n-1})]\, dt \approx [f(y_n) - f(y_{n-1})]\, D(x), \tag{4}$$

where

$$D(x) = \int_0 \frac{\bar{k}(x,t)}{(x-t)^\alpha}\, dt. \tag{5}$$

Substituting (4) in (3), we obtain

$$y_n = \varphi(x) + \int_0^x \frac{\bar{k}(x,t)}{(x-t)^\alpha} f(y_{n-1})\, dt + D(x)\, [f(y_n) - f(y_{n-1})].$$

We construct the successive approximations by the following rule:

$$y_0 - \varphi(x) - D(x) f(y_0) = 0,$$

$$y_n - \varphi(x) - D(x) f(y_n) = \int_0^x \frac{\bar{k}(x,t)}{(x-t)^\alpha} f[y_{n-1}(t)]\, dt - D(x) f(y_{n-1}), \quad (6)$$

$$n = 1, 2, 3, \ldots .$$

Each equation of (6) can be regarded as a transcendental equation for finding y_n.

Theorem

Let $\varphi(x)$ be continuous for $x \geqslant 0$ and $|\varphi(0)| < b$, $b > 0$; $\bar{k}(x, t)$ be continuous in the domain $0 \leqslant t \leqslant x$, $x \geqslant 0$; let a finite $f'(y)$ exist for $-\infty < y < b$;

$$\lim_{y \to b-0} f(y) = +\infty;$$

let $1 - D(x) f'(y) > 0$ for all $x > 0$, $-\infty < y < b$; for $x > 0$ let

$$D(x) = \int_0^x \frac{\bar{k}(x,t)}{(x-t)^\alpha}\, dt \neq 0, \qquad \lim_{y \to -\infty} \frac{f(y)}{y} = 0 \quad \text{or} \quad +\infty.$$

Then: (1) for every n ($n = 0, 1, 2, \ldots$) there exists for all $x \geqslant 0$ a unique continuous function $y_n(x)$, satisfying the corresponding equation of the system (6) and the initial condition $y_n(0) = \varphi(0)$;

(2) $y_n(x) < b$, $x \geqslant 0$;

(3) the sequence of functions $y_n(x)$ converges uniformly in some neighbourhood of zero to the solution of Eq. (1).

The proof of the first two statements, which we will not give, can be carried out by using the theorem on implicit functions [6]. Turning to the proof of the third statement of the theorem, we first show that all the $y_n(x)$ defined by (6) are uniformly bounded in the neighbourhood of the point $x = 0$. Since $y_0(x)$ is continuous and $|y_0(0)| = |\varphi(0)| < b$,

$$|y_0(x)| < M, \qquad |\varphi(0)| < M < b. \quad (7)$$

on some segment $[0, x_1]$.

We write

$$M_1 = \max_{0 \leqslant x \leqslant \delta} y_1(x), \qquad m_1 = \min_{0 \leqslant x \leqslant \delta} y_1(x), \qquad \delta \leqslant x_1,$$

$$m_1 \leqslant \varphi(0) \leqslant M_1, \qquad \Phi(p; q) = \max_{p \leqslant \xi \leqslant q} |f(\xi)|, \; p \leqslant \varphi(0) \leqslant q < b.$$

Then, using (7), we obtain from (6) for $n = 1$

$$\begin{aligned} \varphi(0) &\leqslant M_1 \leqslant \bar{\varphi}(\delta) + \Phi(-M, M)S(\delta) + \bar{D}(\delta)\Phi(m_1, M_1), \\ \varphi(0) &\geqslant m_1 \geqslant -\bar{\varphi}(\delta) - \Phi(-M, M)S(\delta) - \bar{D}(\delta)\Phi(m_1, M_1), \end{aligned} \tag{8}$$

where

$$\bar{\varphi}(\delta) = \max_{0 \leqslant x \leqslant \delta} |\varphi(x)|, \qquad \bar{D}(\delta) = \max_{0 \leqslant x \leqslant \delta} |D(x)|,$$

$$S(\delta) = \max_{0 \leqslant x \leqslant \delta} \left[|D(x)| + \int_0^x \frac{|\bar{k}(x,t)|}{(x-t)^\alpha} dt \right], \qquad \delta \leqslant x_1.$$

We choose $\delta > 0$ so as to satisfy the inequality

$$\bar{\varphi}(\delta) + \Phi(-M, M)S(\delta) + \bar{D}(\delta)\Phi(-|\varphi(0)|, |\varphi(0)|) < M. \tag{9}$$

Then, obviously, we have found a δ_1 satisfying (9), and a q_0 satisfying the condition $|\varphi(0)| < q_0 < M$, such that

$$\bar{\varphi}(\delta_1) + \Phi(-M, M)S(\delta_1) + \bar{D}(\delta_1)\Phi(-q_0, q_0) = q_0, \tag{10}$$

where q_0 is the least positive root for the δ_1 found for the equation

$$\bar{\varphi}(\delta_1) + \Phi(-M, M)S(\delta_1) + \bar{D}(\delta_1)\Phi(-q, q) = q, \qquad |\varphi(0)| \leqslant q < M.$$

The number q_0 will also be the least root of the equation

$$\bar{\varphi}(\delta_1) + \Phi(-M, M)S(\delta_1) + \bar{D}(\delta_1)\Phi(-q_0, q) = q, \quad |\varphi(0)| \leqslant q < M,$$

and the number $-q_0$ the greatest root of the equation

$$-\bar{\varphi}(\delta_1) - \Phi(-M, M)S(\delta_1) - \bar{D}(\delta_1)\Phi(p, q_0) = p, \qquad -M < p \leqslant \varphi(0).$$

We select any $\delta_* < \delta_1$. To simplify the discussion we assume that at least one of the functions $\bar{\varphi}(\delta), S(\delta), \bar{D}(\delta)$ is strictly increasing on $[0, x_1]$. Using the properties of the numbers q_0 and $-q_0$, we find that the inequality

$$q \leqslant \bar{\varphi}(\delta_*) + \Phi(-M, M)S(\delta_*) + \bar{D}(\delta_*)\Phi(-q_0, q) \tag{11}$$

is satisfied in the region $\varphi(0) \leqslant q \leqslant q_k(\delta_*)$, where $q_k > \varphi(0), 0 < q_k(\delta_*) < q_0$, and the inequality

$$q > \bar{\varphi}(\delta_*) + \Phi(-M, M)S(\delta_*) + \bar{D}(\delta_*)\Phi(-q_0, q) \tag{12}$$

is satisfied in the region $q_* < q < M^*$. Here q_* and M^* are certain numbers independent of δ_* satisfying the conditions

$$q_k \leqslant q_* < q_0,\ M^* > q_0,\ M^* \text{ strictly increases with decrease of } \delta_*. \tag{13}$$

On the other hand the inequality

$$p \geqslant -\varphi(\delta_*) - \Phi(-M, M)S(\delta_*) - \bar{D}(\delta_*)\Phi(p, q_0) \tag{14}$$

is satisfied in the region $p_k(\delta_*) \leqslant p \leqslant \varphi(0)$, $p_k < \varphi(0)$, and the inequality

$$p < -\varphi(\delta_*) - \Phi(-M, M)S(\delta_*) - \bar{D}(\delta_*)\Phi(p, q_0) \tag{15}$$

is satisfied in the region $m_* < p < p_*$. Here p_k, m_*, p_* are numbers depending on δ_* which satisfy the conditions

$$m_* < -q_0 < p_* \leqslant p_k < 0,\ m_* \text{ strictly decreases with decrease of } \delta_*. \tag{16}$$

We take any $\delta \leqslant \delta_*$. From (8) we find that $M_1(\delta)$ and $m_1(\delta)$ must satisfy the inequalities

$$\varphi(0) \leqslant M_1 \leqslant \varphi(\delta_*) + \Phi(-M, M)S(\delta_*) + \bar{D}(\delta_*)\Phi(m_1, M_1), \tag{17}$$

$$\varphi(0) \geqslant m_1 \geqslant -\varphi(\delta_*) - \Phi(-M, M)S(\delta_*) - \bar{D}(\delta_*)\Phi(m_1, M_1). \tag{18}$$

We consider the region $\varphi(0) \leqslant M_1 \leqslant q_k$, $\varphi(0) \geqslant m_1 \geqslant p_k$. In view of relations (11) and (14) the inequalities (17), (18) are satisfied in the region considered. In consequence of the same relations (12) and (15) we find in the region $q_* < M_1 \leqslant q_0$, $-q_0 \leqslant m_1 \leqslant p_*$

$$M_1 > \varphi(\delta_*) + \Phi(-M, M)S(\delta_*) + \Phi(-q_0, M_1)\bar{D}(\delta_*) \geqslant \varphi(\delta_*) + \\ + \Phi(-M, M)S(\delta_*) + \bar{D}(\delta_*)\Phi(m_1, M_1), \tag{19}$$

$$m_1 < -\varphi(\delta_*) - \Phi(-M, M)S(\delta_*) - \Phi(m_1, q_0)\bar{D}(\delta_*) \leqslant \\ \leqslant -\varphi(\delta_*) - \Phi(-M, M)S(\delta_*) - \bar{D}(\delta_*)\Phi(m_1, M_1). \tag{20}$$

From the inequalities (17) - (20) and conditions (13) and (16) it follows that

$$\max_{0 \leqslant x \leqslant \delta} |y_1(x)| \leqslant q_0 < M, \qquad \delta < \delta_1 \leqslant x_1. \tag{21}$$

Using (21), it is easy to repeat exactly the same inequalities for $y_2(x)$, $y_3(x)$ etc. Therefore, by induction, we obtain for all n

$$|y_n(x)| < M, \qquad 0 \leqslant x \leqslant \delta < \delta_1, \qquad \delta_1 \leqslant x_1 \tag{22}$$

where δ_1 is found from (10). We prove the uniform convergence of the sequence $\{y_n(x)\}$. Estimating on $[0, x_0]$, $0 < x_0 < \delta_1$, the difference $y_n - y_{n-1}$ from the system (6) and using (22), we obtain

$$|y_n - y_{n-1}| \leqslant R \max_{0\leqslant x\leqslant x_0} |y_{n-1} - y_{n-2}|\, S(x_0) + R\bar{D}(x_0)\, |y_n - y_{n-1}|, \qquad n \geqslant 2, \tag{23}$$

$$|y_1 - y_0| \leqslant MRS(x_0) + R\bar{D}(x_0)\, |y_1 - y_0|,$$

where

$$R = \max_{-M\leqslant\xi\leqslant M} |f'(\xi)|.$$

Choosing x_0 so small that

$$1 - R(M)\bar{D}(x_0) > 0, \tag{24}$$

we obtain from (23)

$$|y_n - y_{n-1}| \leqslant \frac{RS(x_0)}{1 - R\bar{D}(x_0)} \max_{0\leqslant x\leqslant x_0} |y_{n-1} - y_{n-2}|, \qquad n \geqslant 2,$$

$$|y_1 - y_0| \leqslant M \frac{RS(x_0)}{1 - R\bar{D}(x_0)}. \tag{25}$$

Using (25) successively for $n = 2, 3, \ldots,$ we find

$$|y_n - y_{n-1}| \leqslant M \left[\frac{RS(x_0)}{1 - R\bar{D}(x_0)}\right]^n, \qquad n = 1, 2, \ldots . \tag{26}$$

For

$$\frac{RS(x_0)}{1 - R\bar{D}(x_0)} < 1 \tag{27}$$

the sequence $\{y_n(x)\}$ converges uniformly on $[0, x_0]$ to some continuous function $Y(x)$, which, as is easily verified, satisfies Eq. (1). The uniqueness of the solution satisfying the condition $y < b$, $x \geqslant 0$, is proved in [7].

It must be emphasised that for the successive approximations of the solution of Eq. (1) with an $f(y)$ of the type (2), constructed by the method of averaging the functional corrections [3], there is no prior guarantee of their existence for all $x \geqslant 0$.

The theorem presented above is also proved under similar conditions for functions $f(y)$ defined for all y.

It is necessary to mention that it is not obligatory to choose $D(x)$ in the form (5). If we take as $D(x)$ any continuous function equal to zero for $x = 0$ and satisfying the conditions of the theorem, all its results remain true. This provides the possibility of choosing $D(x)$ for the more efficient construction of the y_n. The form of $D(x)$ indicated in (5) gives completely satisfactory approximations in the class of problems which interest us. We mention that in a number of cases Eqs. (6) enable us to find y_n in explicit form, for example, if $f(y) = y/(b - y)$.

We also point out that in practice the successive approximations obtained from (6) converge over a considerably greater region than by the estimates (10), (24) and (27). For a number of problems it is possible to prove that all the y_n have at infinity the asymptotic behaviour of the exact solution. It is therefore to be expected that their difference from the exact solution will also be small for all $x \geqslant 0$. This is confirmed in those cases where the exact solution is known.

If the function $f(y)$ is linear, $f(y) = y$, then for $1 - D(x) > 0$, $x \geqslant 0$, all the y_n are found explicitly from (6). As an example we give a graph

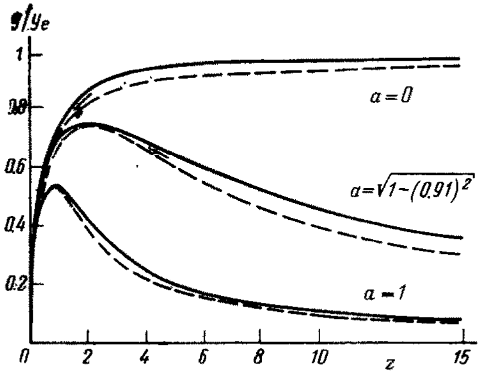

FIG. 1

———— exact value y/y_e

– – – – – null approximation y_0/y_e

(see Fig. 1) of the exact solution (continuous curve) and the null approximation (dashed curve) of the linear equation

$$y = \frac{2}{\sqrt{(\pi u)}} w(\sqrt{(ux)}) + \frac{2}{y_e\sqrt{(\pi u)}} \int_0^x \left[\frac{d}{d\xi} w(\sqrt{[u(x-\xi)]}) \right] y(\xi)\, d\xi,$$

$$u = \text{const}, \qquad y_e = \text{const}, \quad u \geqslant 0, \qquad 0 < y_e < 1,$$

$$w(x) = \exp(-x^2) \int_0^x e^{t^2}\, dt,$$

arising in the problem of an electrochemical reaction with an adsorption depolariser [8];

$$\frac{y_0(x)}{y_e} = \frac{w(az)}{w(az) + a\sqrt{\pi}/4}, \qquad a = 2y_e\sqrt{u}, \qquad z = \frac{\sqrt{x}}{2y_e},$$

$$\frac{y_{\text{точ}}}{y_e} = \frac{1}{\sqrt{(1-a^2)}} \{\exp[(1-\sqrt{(1-a^2)})^2 z^2] \operatorname{erfc}[(1-\sqrt{(1-a^2)})z] - \exp[(1+\sqrt{(1-a^2)})^2 z^2] \operatorname{erfc}[(1+\sqrt{(1-a^2)})z]\} \quad \text{for} \quad a \geqslant 0,\ a \neq 1,$$

$$\frac{y_{\text{точ}}}{y_e} = \frac{4z}{\sqrt{\pi}} [1 - \sqrt{\pi}\, z \exp(z^2) \operatorname{erfc}(z)] \quad \text{for} \quad a = 1. \tag{28}$$

If $a = 0$ the next approximation also is expressed very simply in terms of the elementary functions:

$$\frac{y_1}{y_e} = \left(\frac{4z/\sqrt{\pi}}{1+(4/\sqrt{\pi})z}\right)^2 + \frac{\pi/2 - [(4z/\sqrt{\pi})^2-1]^{-1/2}\ln[4z/\sqrt{\pi}+\sqrt{[(4z/\sqrt{\pi})^2-1]}]}{1+(4/\sqrt{\pi})z} \quad \text{for} \quad z \geqslant \frac{\sqrt{\pi}}{4},$$

$$\frac{y_1}{y_e} = \frac{\pi}{2} - \frac{2}{\sqrt{[1-(4z/\sqrt{\pi})^2]}}\left[\operatorname{arctg}\frac{1+4z/\sqrt{\pi}}{\sqrt{[1-(4z/\sqrt{\pi})^2]}} - \operatorname{arctg}\frac{4z/\sqrt{\pi}}{\sqrt{[1-(4z/\sqrt{\pi})^2]}}\right] \quad \text{for} \quad z < \frac{\sqrt{\pi}}{4}. \tag{29}$$

Using (28), (29), it is possible to estimate the accuracy of the second approximation:

$$\left|\frac{y_{\text{точ}} - y_0}{y_{\text{точ}}}\right| 100\% < 8\%, \quad \left|\frac{y_{\text{точ}} - y_1}{y_{\text{точ}}}\right| 100\% < 1\%, \quad z \geqslant 0.$$

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