LETTERS

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Nonlinear saturation of Rayleigh–Taylor instability in thin films

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A general mechanism of nonlinear saturation of instabilities in flowing films is described using the Rayleigh–Taylor instability as an example. The combined action of flow shear and surface tension is the essence of the saturation mechanism. As a result, the streamwise perturbations of the interface that would rupture a stagnant film do not rupture a film flowing in a certain range of shear rates.

We have studied the stability of liquid films *flowing* under heavier fluids, with respect to disturbances independent of the spanwise coordinate. Such a film, if stagnant, would rupture as a consequence of the Rayleigh–Taylor instability.[1-3] Consider a film flowing not too fast [i.e., with a Reynolds number small in a sense specified below, see Eq. (10)]. Its *linear* stability properties do not differ essentially from those in the stagnant case. Longwave perturbations of small amplitudes do grow (exponentially) at first. However, we find that the flowing films (in a certain range of shear rates) can sustain without rupture the perturbations which would break up the films in stagnant situations. The (Rayleigh–Taylor) instability of the interface saturates at some level lower than the unperturbed film thickness. This is a result of a *nonlinear* flow-induced and surface-tension-assisted mechanism described below which occurs in a variety of film flows with different destabilizing factors.

Consider for simplicity the case of (two incompressible) fluids with equal viscosities $\mu_F = \mu_L \equiv \mu$, where μ_F relates to the lower fluid ("film") and μ_L to the upper one ("liquid"). The liquid (of thickness $h_L \equiv H$) is assumed to be thicker than the film (of thickness $h_F \equiv h$),

$$h \ll H, \tag{1}$$

and more dense,

$$\rho_L > \rho_F. \tag{2}$$

We choose the reference frame of the unperturbed interface. The interface occupies the plane $y = 0$ in our rectangular Cartesian coordinates. The film is in no-slip contact with a solid plate below it, at $y = -h$. The plate moves in the x direction at a (constant) velocity U_b. The constant gravitational acceleration is denoted as g.

In the primary velocity distribution then the vertical components are $V_F = V_L = 0$. The horizontal ones U_F and U_L are x directed and vary only in the vertical direction:

$$U_F = Wy, \quad U_L = Wy, \tag{3}$$

where

$$W \equiv U_b/h = U_a/H. \tag{4}$$

Here U_a is the upper plate velocity, $U_a = U_L(H)$.

Consider the equations and boundary conditions for the (streamwise) perturbations of the velocities $u_{F,L}(t, x, y)$ and $v_{F,L}(t, x, y)$, the pressures $p_{F,L}(t, x, y)$, and the interface elevation $\eta(t, x)$. We restrict consideration to primary flows which satisfy a number of conditions: some nondimensional groups of their parameters (which are specified below) have to be small. This allows us to neglect certain terms in the perturbation problem in order to arrive at a closed-form asymptotic equation for the interface evolution.

Specifically, the assumptions are as follows. We consider thin films, in the sense that the ratio κ of the film disturbance y scale to the x scale is small,

$$\kappa \equiv h/L \ll 1, \tag{5}$$

where L is the characteristic perturbation x scale of our problem; we will see below that it can be expressed in terms of the unperturbed system parameters as

$$L \equiv [\sigma/g(\rho_L - \rho_F)]^{1/2} \equiv (\sigma/\delta)^{1/2}, \tag{6}$$

where σ is the surface tension, $\delta \equiv g(\rho_L - \rho_F)$. [It will be shown below that L is of order of the critical wavelength for infinitesimally small perturbations. The shorter-scale perturbations die out, while the longer-scale perturbations are (linearly) unstable—and thus subject for a nonlinear theory.] The next condition (which guarantees that the film y scale is much shorter than that of the liquid) was actually stated earlier, Eq. (1):

$$\beta \equiv h/H \ll 1. \tag{7}$$

The next two small parameters, designated as ϵ and α, are needed inasmuch as we would like to neglect the time derivative terms in the perturbation Navier–Stokes equations and the "inertia" terms (and thereby retain only the viscous and pressure terms). The parameter ϵ has the meaning of an effective Strouhal number for the film perturbations (which is a measure of the ratio of the time-derivative terms to the inertial ones): it can be defined as

$$\epsilon \equiv L/TU_b \ll 1. \tag{8}$$

Here T is the time scale of the problem, which can be expressed, as will be shown below, as

$$T \equiv \mu\sigma/h^3\delta^2. \tag{9}$$

[As a matter of fact, the time-scale expression (9) follows already from the linear perturbation theory.] Finally, α is essentially the film perturbation Reynolds number and can be defined as

$$\alpha \equiv U_b h^2/L\nu \ll 1, \tag{10}$$

where $\nu \equiv \mu/\rho_F$.

In addition we make some assumptions as to the magnitude of the perturbation solutions, which can be justified *a posteriori* (see also Ref. 4). First, the interface elevation η is assumed to be small in comparison with the film thickness h:

$$\eta/h \ll 1. \tag{11}$$

Second, the perturbation velocity u_F is assumed to be small in the following sense:

$$\frac{u_F}{U_b} \sim \frac{u_L}{U_b} \ll \frac{\eta}{h}. \tag{12}$$

Using the conditions (5)–(12), many terms in the (generally speaking, nonlinear) perturbation equations can be estimated to be negligible to leading order. [For example, the condition (5) allows one to neglect the x derivatives in the film in comparison to their y counterparts.]

As a result, to leading order, the asymptotic perturbation problem assumes the following form.

The film Navier–Stokes equations are (in what is, in effect, the "lubrication approximation"[5] and the Stokes approximation combined)

$$\frac{\partial p_F}{\partial y} = 0, \tag{13}$$

and

$$\frac{\partial^2 u_F}{\partial y^2} = \frac{1}{\mu}\frac{\partial p_F}{\partial x}. \tag{14}$$

The (exact) incompressibility equation is

$$\frac{\partial u_F}{\partial x} + \frac{\partial v_F}{\partial y} = 0. \tag{15}$$

The interface conditions assume the form

$$\frac{\partial u_F}{\partial y} = 0 \quad (y = 0) \tag{16}$$

(tangential stress condition),

$$p_F = -\delta\eta - \sigma\frac{\partial^2\eta}{\partial x^2} \tag{17}$$

(normal stress condition), and

$$\frac{\partial\eta}{\partial t} + \frac{U_b}{h}\eta\frac{\partial\eta}{\partial x} = v_F\Big|_{y=0} \tag{18}$$

(kinematic condition).

[In Eq. (18), we have used the following approximations for $U_F(\eta)$ and $v_F|_{y=\eta}$: $U_F(\eta) \approx U_F(0) + (\partial U_F/\partial y)(0)\eta = (U_b/h)\eta$, and $v_F|_{y=\eta} \approx v_F|_{y=0}$; and we have neglected the term $(u_F|_{y=\eta})(\partial\eta/\partial x) \approx (u_F|_{y=0})(\partial\eta/\partial x)$ in comparison with the second term in Eq. (18), by virtue of Eq. (12).] Also, we make use of the following boundary conditions at the lower plate:

$$u_F = 0 \quad (y = -h), \tag{19}$$

and

$$v_F = 0 \quad (y = -h) \tag{20}$$

(the upper-plate counterparts are $u_L = v_L = 0, y = H$).

Having formulated the problem (13)–(20), one can proceed as follows to express v_F in Eq. (18) in terms of η (and thus obtain the equation for the interface elevation in closed form).

It is easy to find from Eqs. (15) and (20) the expression

$$v_F|_{y=0} = -\int_{-h}^{0}\frac{\partial u_F}{\partial x}dy. \tag{21}$$

One can then express u_F in terms of p_F by solving Eq. (14) with the two boundary conditions, Eqs. (19) and (16), and making use of Eq. (13). Substituting this solution into Eq. (21), v_F is expressed in terms of p_F. With this expression, Eq. (18) gives

$$\frac{\partial\eta}{\partial t} + W\eta\frac{\partial\eta}{\partial x} - \frac{h^3}{3\mu}\frac{\partial^2 p_F}{\partial x^2} = 0. \tag{22}$$

Since p_F is already given in terms of η by Eq. (17), we arrive at the (central for this work) equation[6] for the interface evolution:

$$\frac{\partial\eta}{\partial t} + W\eta\frac{\partial\eta}{\partial x} + \frac{h^3}{3\mu}\left(\delta\frac{\partial^2\eta}{\partial x^2} + \sigma\frac{\partial^4\eta}{\partial x^4}\right) = 0. \tag{23}$$

Let us analyze the interface evolution described by this equation. When η is very small, the nonlinear term is negligible. The linear equation

$$\frac{\partial\eta}{\partial t} + \frac{h^3}{3\mu}\left(\delta\frac{\partial^2\eta}{\partial x^2} + \sigma\frac{\partial^4\eta}{\partial x^4}\right) = 0 \tag{24}$$

leads to the dispersion relation

$$\omega = i(h^3/3\mu)k^2(\delta - \sigma k^2), \tag{25}$$

for linear modes $\eta \propto e^{-i\omega t + ikx}$. It shows the Rayleigh–Taylor instability for the long waves $k^2 < \delta/\sigma$. So the characteristic length L, Eq. (6), is of the order of the critical wavelength. Equation (25) shows the growth-rate maximum at a wavelength of the same order as L and the inverse maximal growth rate is of the order of T, Eq. (9). But if η were to grow—due to the destabilizing factor [δ term in Eq. (24)]—beyond $h\epsilon$, i.e., if $\eta \gg h\epsilon$, then the linear terms in Eq. (23) could be neglected, as one can show easily. This step results in the well-known[7] nonlinear equation

$$\frac{\partial\eta}{\partial t} + W\eta\frac{\partial\eta}{\partial x} = 0. \tag{26}$$

The process it describes is the distortion of the "wave profile" $\eta(t, x)$. It can be expressed in the following words (in our "interface interpretation" of the equation): every point of the interface moves in the (horizontal) x direction with velocity proportional to that point's elevation η. Thus, the points where $\eta = 0$ ("zeroes") do not move, while the points of maximal elevation move faster than all other points. Let the initial interface profile be symmetric in the interval between two neighboring zeroes [e.g., $\eta(0,x) \propto \sin(x/l)$, $0 \leqslant x \leqslant (\pi/2)l$, $l \gtrsim L$]. In the subsequent evolution the symmetry is lost since the maximum moves farther from one of the zeroes and closer to the other one. The latter would have been passed in a finite interval of time. However, in our case this process of steepening of the forward faces of the interface profile does not result in the eventual breakup, because the surface-tension term in Eq. (23), which was negligible initially, becomes important when the steepening process has advanced sufficiently far. Indeed, as was mentioned above, the elevations do not change during the process of steepening described by Eq. (26). In contrast, the interface curvatures grow, i.e., the effective x scales contract. Hence, the surface-tension term containing the highest-order x derivative grows faster than other terms. Due to the enhanced action of surface tension, excessive elevations are reduced, the steepening process is slowed down and later reversed. But if the elevations become small ($\eta \ll h\epsilon$), then gravity [δ term in Eq. (23)] prevails and the elevations grow again. As a result of the combined action of these three factors—destabilizing gravity, flow-induced scale contractions, and stabilizing surface tension—finite-amplitude oscillations set in, such that all the terms in Eq. (23) are of the same order. This condition yields the time scale T [Eq. (9)], x scale L [Eq. (6)], and a small elevation η, $\eta/h \sim \epsilon$ [cf. Eqs. (11) and (8)].

These conclusions from physical analysis are consistent with the results of the numerical investigation[8] of a similar equation.

We would like to emphasize that the physical interpretation of the saturation mechanism given here is applicable to a wide class of film flows. The saturation mechanism does not depend on the nature of the initial destabilizing influence (here gravity) although it demands that some destabilizing factor be present. The same process of shear steepening of the forward faces of interface profiles—that generates large curvatures and hence, through surface tension, significant pressure perturbations causing saturation—can occur in a wide variety of flows. Indeed, the same mechanism can be found in free-surface film flows down planes and cylindrical surfaces[8,9] (although those authors[8,9] consider neither the nature of saturation nor the question of film rupture), where gravity is destabilizing. For the plane "liquid-film" systems like those considered in the present work but with fluids of nonequal viscosities in Poiseuille-type flows, there is destabilization due to the difference of viscosities as such,[10] and the saturation mechanism is present as well. We intend to consider this case in detail elsewhere, as well as two-layer Poiseuille-type film flows inside cylindrical capillaries. In the latter case it is interesting that the surface tension plays two contrasting roles: on one hand, it is a linearly destabilizing factor and on the other hand, it takes a stabilizing part in the nonlinear saturation mechanism. Elsewhere[4] we find the same mechanism in very thin films $h \lesssim 10^{-4}$ cm, where the van der Waals molecular forces take over the destabilizing role which gravity plays in the Rayleigh–Taylor instability.

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