Annular Flows Can Keep Unstable Films from Breakup: Nonlinear Saturation of Capillary Instability

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We consider a fluid film on the inner walls of a capillary. The film surrounds another fluid in the core. It is known that the capillary instability, driven by the surface tension at the fluid–fluid interface, breaks up the film if it is primarily stagnant. In contrast, as we show, a primary flow, in a certain range of parameters, can keep the linearly unstable film from rupturing. This is a result of the nonlinear low-level saturation of the interface instability. This saturation is due to the coordinated action of the destabilizing factors, the shear of flow, and the surface tension at the interface. The resulting state of the interface is, in general, chaotic oscillations, with their amplitude being much less than the unperturbed film thickness. The approximate equation of interface evolution is derived. The saturation mechanism is explained. The characteristic scales of the developed oscillations are found, and the parameter range of the theory applicability is discussed.

INTRODUCTION

Fluid films on inner walls of tubes occur in nature, industry, and laboratory. Examples abound. In laboratory practices, a capillary can be washed clean of a fluid by displacing it with another fluid, and in this process the initially present fluid is often left behind as a thin film on the capillary walls (1). Also, water and oil can displace one another in a porous rock.

In crude oil pipelines, the water entrained in the oil can settle out to the pipe walls as a thin film. On the other hand, one can think of providing such a film artificially to reduce the power needed for the oil transportation [see, e.g., Ref. (2)].

In coating, if the inner wall of a tube is to be coated with a film of a material with certain desirable properties, one could consider depositing the material in liquid state, provided the capillary instability would not break up the film.

The stability of such tube wall films, generally surrounding another fluid in the core, is obviously of a significant interest. The linear analysis shows a long-wave instability (3) and gives an order estimate of the film breakup time. The nonlinear process of the film breakup under the (capillary) instability in two-fluid tubes with *no primary flow* was studied recently (4).

In the present communication, we show that primary flows can play a crucial role of keeping annular films from rupturing. The capillary instability saturates long before the interface deviation can become comparable with the film thickness. This is a result of a nonlinear mechanism combining the factors destabilizing the fluid–fluid interface, its nonlinear distortion by the flow shear, and the restoring action of surface tension. This mechanism was first described (5) in the example of Rayleigh–Taylor instability in a planar two-fluid system. For such a system with no pri-

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mary flow, the nonlinear breakup of the film has also been studied recently (6). Thus, the present work which demonstrates a qualitatively new effect of nonlinearity in capillary instability, its saturation in flowing films, is to the nonlinear study (4) of primarily stagnant annular films, approximately as the work (5) is to (6) for planar films.

A related study of outer films flowing down vertical cylindrical columns was done in Ref. (7).

PRIMARY FLOW

We consider a concentric flow of two fluids along a straight circular capillary of radius c, as sketched in Fig. 1. The outer fluid is a thin film:

$$h \equiv c - b \ll b, \qquad [1]$$

where h is the film thickness and b the radius of the unperturbed fluid–fluid interface.

To isolate the surface-tension-driven "capillary instability" intrinsic to the cylindrical interface (8), we assume the (incompressible) fluids to be of equal densities ρ and viscosities μ. The capillary is assumed to be sufficiently narrow, so that gravity can be neglected as discussed in some detail for the cylindrical geometry in Ref. (4). The corresponding small parameter, a modified Bond number, is cited below (see Eqs. [47] and [48]).

The primary flow is driven by the constant gradient of pressure P,

$$\frac{\partial P}{\partial z} = cst \equiv A, \quad \frac{\partial P}{\partial r} = \frac{\partial P}{\partial \phi} = 0, \qquad [2]$$

where z, ϕ, and r are the usual cylindrical coordinates, with $r = 0$ on the capillary axis (see Fig. 1). The flow velocity has the well-known Poiseuille profile

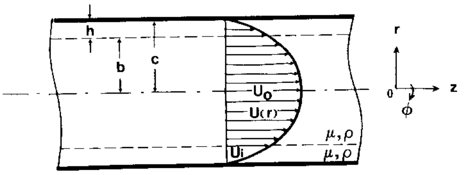

FIG. 1. Definition sketch for the primary flow.

$$U = \frac{A}{4\mu}(c^2 - r^2), \quad V = W = 0, \qquad [3]$$

(where U, V, and W are, correspondingly, the z-, ϕ-, and r-components of velocity) so that only the axial velocity is nonzero, and even it does not depend on z and ϕ. It is easy to check that Eqs. [2] and [3] satisfy all the fluid equations and boundary conditions [see, e.g., Ref. (3)].

INTERFACE EVOLUTION EQUATION

The exact perturbation equations and relevant boundary conditions are given in the Appendix. To obtain the interface evolution equation, we will neglect certain terms in the perturbation equations. Such a procedure is justified if one can show that substituting the solution of the simplified equations into the full equations renders the neglected terms indeed relatively small. In the Appendix we demonstrate such self-consistency for our derivation assumptions, in a specified range of system parameters.

First, we assume the combination of the lubrication approximation [see, e.g., Ref. (9)] and the Stokes approximation for the film, neglecting the inertia and time-derivative terms in the Navier–Stokes equations, and expecting the z and ϕ scales to be of the order of b which is much greater than the radial scale h, (see Eq. [1]):

$$\kappa \equiv \frac{h}{b} \ll 1. \qquad [4]$$

The Navier–Stokes equations (see Appendix), yield in the main approximation

$$\frac{\partial p}{\partial z} = \mu \frac{\partial^2 u}{\partial r^2}, \qquad [5]$$

$$\frac{\partial p}{r\partial \phi} = \mu \frac{\partial^2 v}{\partial r^2}, \qquad [6]$$

and

$$\frac{\partial p}{\partial r} = 0, \qquad [7]$$

where u, v, w, and p are, correspondingly, the perturbations of z-, ϕ-, r-velocity, and of the pressure, in the film. Using the Navier–Stokes equations for the core fluid and the normal-stress condition at the interface, one finds (see Appendix) in the main approximation

$$p = -\frac{\sigma}{b^2}\eta - \sigma\Delta\eta, \qquad [8]$$

where $\eta(t, z, \phi)$ is the film thickening,

$$\eta \equiv b - r_i, \qquad [9]$$

if the interface is described by

$$r = r_i(t, z, \phi). \qquad [10]$$

In [8], σ is the surface tension constant and Δ is the two-dimensional Laplacian,

$$\Delta \equiv \frac{\partial^2}{\partial z^2} + \frac{1}{b^2}\frac{\partial^2}{\partial \phi^2}. \qquad [11]$$

In particular, for the contribution of the inertial terms into the pressure perturbations of the core to be negligible as compared with the film pressure perturbations, one needs, as shown in the Appendix, the parameter condition

$$R \equiv \frac{U_i h}{\nu} \ll 1, \qquad [12]$$

where U_i is the interface velocity,

$$U_i \equiv U(b), \qquad [13]$$

and ν is the kinematic viscosity,

$$\nu \equiv \mu/\rho. \qquad [14]$$

Obviously, R is the film Reynolds number based on the thickness h and the interface velocity U_i. The condition [12] can be easily seen to be stronger than the one providing for the smallness of inertial terms compared to viscous terms in the film Navier–Stokes equations for perturbations (see Appendix). The conditions [4], [12], and an additional condition which provides, in particular, for neglecting the time-derivative terms in the Navier–Stokes equations (and whose explicit form is given below) suffice for justifying all the approximation equations, including [5] through [8]. We will use the continuity equation [A5] approximated as

$$\frac{\partial u}{\partial z} + \frac{1}{b}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial r} = 0. \qquad [15]$$

On the interface, the kinematic condition of impermeability [A7] can be written as

$$\frac{\partial \eta}{\partial t} + \tilde{U}\frac{\partial \eta}{\partial z} + u\frac{\partial \eta}{\partial z} + \frac{v}{r}\frac{\partial \eta}{\partial \phi} + w = 0 \qquad [16]$$

in the reference frame of the unperturbed interface, where $\tilde{U}$ is the primary flow profile in that frame:

$$\tilde{U}(r) = U - U(b). \qquad [17]$$

Thus, $\tilde{U}(b) = 0$, and $\tilde{U}$ on the interface in (16) can be approximated as

$$\tilde{U}(r_i) = \tilde{U}(b) + \frac{\partial \tilde{U}}{\partial r}(r_i - b) = \frac{Ab}{2\mu}\eta \approx \frac{U_i}{h}\eta \qquad [18]$$

assuming that the film thickening is small,

$$\eta/h \ll 1. \qquad [19]$$

We also assume that the third term in [16] can be neglected,

$$u(b) \ll \frac{U_i}{h}\eta. \qquad [20]$$

Similarly, the fourth term in [16] is assumed to be negligible. As a result, [16] takes the form

$$\frac{\partial \eta}{\partial t} + G\eta\frac{\partial \eta}{\partial z} + w = 0 \qquad [21]$$

where

$$G \equiv \frac{U_i}{h} \qquad [22]$$

and w can be taken at $r = b$.

We are going to express w in [21] in terms of η to obtain a closed equation for the interface evolution. This can be done as follows. From [5], [6], and [7] we can find u and v in terms of p. Substituting these in [15] will lead to expressing w in terms of p and then, through [8], in terms of η.

In implementing this plan, one finds from [5]

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial z}[r^2 - c^2 - 2b(r - c)], \qquad [23]$$

where we have used the no-slip conditions $u(c) = 0$, as well as the tangential stress condition [A8] reduced to (see Appendix)

$$\frac{\partial u}{\partial r} = 0, \quad (r = b). \qquad [24]$$

Analogously, from (6),

$$v = \frac{1}{2\mu} \frac{\partial p}{b \partial \phi} [r^2 - c^2 - 2b(r - c)]. \qquad [25]$$

From [15],

$$\frac{\partial w}{\partial r} = -\left(\frac{\partial u}{\partial z} + \frac{1}{b} \frac{\partial v}{\partial \phi}\right). \qquad [26]$$

By integrating, one obtains

$$w = -\int_c^b \left(\frac{\partial u}{\partial z} + \frac{1}{b} \frac{\partial v}{\partial \phi}\right) dr, \qquad [27]$$

since $w = 0$ at $r = c$. Substituting now [23] and [25] into [27] leads to

$$w = -\frac{1}{2\mu} \Delta p \int_c^b [r^2 - c^2 - 2b(r - c)] dr \qquad [28]$$

(where Δ is defined in [11]). The integral in [28] is calculated easily. This leads to

$$w = -\frac{h^3}{3\mu} \Delta p, \qquad [29]$$

which is transformed with [8] to

$$w = \frac{h^3 \sigma}{3\mu} \left(\frac{1}{b^2} \Delta \eta + \Delta^2 \eta\right). \qquad [30]$$

Finally, by substituting [30] into [21], we arrive at the interface evolution equation

$$\frac{\partial \eta}{\partial t} + \frac{U_i}{h} \eta \frac{\partial \eta}{\partial z} + \frac{h^3 \sigma}{3\mu} \left(\frac{1}{b^2} \Delta \eta + \Delta^2 \eta\right) = 0. \qquad [31]$$

We recall the notations here: h is the film thickness, η the film thickening, U_i the interface velocity (or the modulus of the capillary wall velocity in the interface reference frame) proportional to the driving pressure gradient (see [13], [3], and [2]), b is the constant radius of the unperturbed interface, and Δ is the two-dimensional Laplacian (11).

LINEAR INSTABILITY STAGE

When the film thickening η is infinitesimally small, the nonlinear term in [31] is negligible:

$$\frac{\partial \eta}{\partial t} + D^2 \left(\frac{1}{b^2} \Delta + \Delta^2\right) \eta = 0, \qquad [32]$$

where

$$D^2 \equiv \frac{\sigma h^3}{3\mu}. \qquad [33]$$

This leads to the dispersion equation expressing, for a linear mode

$$\eta \propto e^{\alpha t} e^{i(kz + m\phi)}, \quad (m = 0, \pm 1, \pm 2, \cdots)$$

the growth rate α as a function of the wavenumbers k and m:

$$\alpha = D^2 \left(k^2 + \frac{m^2}{b^2}\right) \left[\frac{1}{b^2} - \left(k^2 + \frac{m^2}{b^2}\right)\right]. \qquad [34]$$

For nonzero m, the square bracket in [34] is clearly nonpositive (and strictly negative if $k \neq 0$ in addition to $m \neq 0$). Thus, the instability, $\alpha > 0$, is present for only axisymmetric modes, $m = 0$, or

$$\eta \propto e^{ikz}. \qquad [35]$$

The instability is longwave,

$$|k| < b^{-1}. \qquad [36]$$

The wavenumber k_m at which the growth rate α takes its maximal value α_m is easily found to be of the order of b^{-1} as well. Thus, b is the characteristic length scale L of this capillary instability:

$$L = b. \qquad [37]$$

The characteristic time scale, the inverse maximal growth rate α_m^{-1}, is found to be of order T,

$$T \equiv \frac{\mu b^4}{h^3 \sigma}, \qquad [38]$$

by substituting $k = b^{-1}$ (and $m = 0$) in [34].

NONLINEAR SATURATION MECHANISM

The interface evolution equation [31] has the same structure as its analog for the case of Rayleigh–Taylor instability of a planar two-

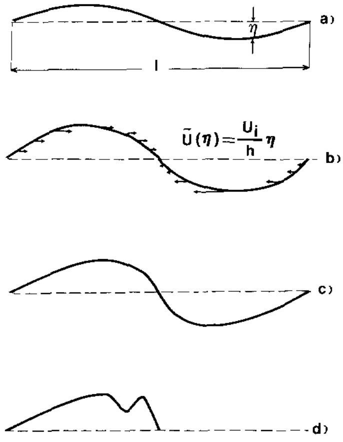

FIG. 2. Sketch for the mechanism of instability saturation: (a) small initial interface deviation, $\eta \propto \sin z$; (b) the instability-increased deviation of the interface, and the primary velocity profile in the reference frame of unperturbed interface; (c) the interface distortion by the flow shear leading to larger local curvatures; (d) the restoring action of surface tension showing in regions of larger interface curvatures.

fluid flow (5). The general mechanism of nonlinear saturation of instabilities in flowing films, described in (5), applies here. Let us discuss it for the case of capillary instability.

Consider a linear mode of wavelength l (Fig. 2a). It will grow if $l > b$ (see Fig. 2b). The destabilizing, second-derivative term in [31] is greater than the stabilizing fourth-derivative term as long as the linear stage goes on. However, the linear stage leads to its own end since the nonlinear term increases the fastest with increasing η. It becomes comparable with the destabilizing term at

$$\eta \sim \frac{D^2 h}{b^2 U_i l}, \qquad [39]$$

or, using [38],

$$\frac{\eta}{h} \sim \frac{b^2}{l T U_i} < \frac{b}{T U_i}. \qquad [40]$$

We assume the parameter condition

$$\epsilon \equiv \frac{b}{T U_i} = \frac{L}{T U_i} \ll 1. \qquad [41]$$

If η grew still further, the nonlinear term would exceed the linear ones. Neglecting the latter, we come to the equation

$$\frac{\partial \eta}{\partial t} + G\eta \frac{\partial \eta}{\partial z} = 0. \qquad [42]$$

This is the well-known "simplest hyperbolic" equation (10). It describes the distortion of the interface profile (see Figs. 2b–c). In the reference frame of the unperturbed interface, every point of (perturbed) interface moves with velocity proportional to the interface deviation η. This leads to steepening the forward faces of interface profile as in Fig. 2c. Thus regions of higher curvature, i.e., of shortened length scale, are created. In this scale-shortening, the fourth-derivative term becomes dominant leading to the local decrease of η as shown in Fig. 2d. This can give rise to other high-curvature regions, etc. One can not follow the further evolution in much detail. In general, it is chaotic oscillations of the interface, as has been shown numerically and analytically (11). Some computer simulation results are shown in Fig. 3 for the "canonical" rescaled version of [31],

$$\frac{\partial \Phi}{\partial \tau} + \Phi \frac{\partial \Phi}{\partial \xi} + \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^4 \Phi}{\partial \xi^4} = 0. \qquad [43]$$

The characteristic scales of the developed interface oscillations are determined by the condition that, as we have seen, all terms in [31] should be of the same order. The order equality of the linear terms of characteristic length scale L_n gives

$$\frac{\eta}{b^2 L_n^2} \sim \frac{\eta}{L_n^4},$$

whence $L_n \sim b$. So, the nonlinear length scale L_n coincides with the linear one [37]:

$$L_n = L = b. \qquad [44]$$

Analogously, the characteristic time scale is the same as that [38] for the linear stage of instability. This conclusion follows from the order equality of the time-derivative and linear z-derivative terms in [31]. Finally, by equalizing the orders of nonlinear and linear terms one finds that the characteristic amplitude of

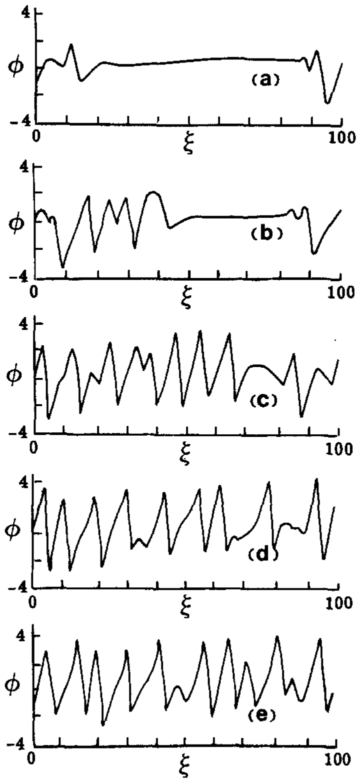

FIG. 3. The result of the numerical solution of Eq. [43] by Sivashinsky and Michelson (11).

the oscillations is small (which is consistent with [19]):

$$\frac{\eta}{h} \sim \epsilon \qquad [45]$$

where ϵ is defined in [41].

In the capillary instability case considered here, the general saturation mechanism (5) of flowing-film instabilities has some interesting characteristics. First, surface tension plays two contrasting roles. It linearly destabilizes long-wave perturbations as far as they are sufficiently small, but on the other hand, when the nonlinear stage sets in, surface tension is an important factor in saturation of instability, decreasing those interface deviations which are too large. Second, and most remarkably, there is no unstable spanwise perturbations. Such interface perturbations, which are not distorted by flow shear, could not saturate and would lead to the film breakup (5). In capillary instability, a primary flow keeps an unstable film from breakup no matter what the initial perturbations are. On the contrary, a stagnant film will inevitably break up, as was studied in (4).

DISCUSSION

The domain of applicability of our conclusions can be expressed in terms of the interface velocity U_i by combining [41] and [12]:

$$\frac{\sigma h^3}{\mu b^3} \ll U_i \ll \frac{\nu}{h}. \qquad [46]$$

(Of course, the condition [1], stating that the film is thin, is implied.) The first inequality in [46] can be interpreted as the smallness of the so-called Strouhal number, the ratio of the time-derivative to the inertia terms in the Navier–Stokes equations for film perturbations (see Appendix). It is necessary that the Strouhal number be small for the film to be kept from breakup.

The second inequality requires that the film Reynolds number be small. It is merely sufficient but not necessary for our conclusions. Quite possibly, the film does not rupture beyond the upper limit of the domain [46]. However, it is physically clear that at very large Reynolds numbers the film will be ruptured, e.g., by turbulence.

The conclusion that the film does not break up over a limited interval of velocities is in accordance with experimental observations (2), although strictly speaking the parameters there were out of the applicability domain of the present theory. In particular, the ratio of the film thickness to the tube radius was not sufficiently small in those experiments, in violation of our applicability condition [1]. Also, gravity was not negligible in (2). Indeed, the condition for gravitational effects to be neglected is (4)

$$B \ll h/c, \qquad [47]$$

where B, the so-called Bond number measuring the relative importance of gravity over surface tension, is defined as

$$B \equiv \frac{(\Delta\rho)gc^2}{\sigma}. \qquad [48]$$

The tube radius was relatively large, ~ 1 cm, in the experiments (2), and even though the density difference $\Delta\rho$ was small, $\sim 10^{-2}$ g cm^{-3}, the condition [47] was violated.

The following parameter values (in CGS units) are a numerical example of situations when our theory is well applicable: $b \sim c \sim 10^{-1}$, with $h \sim 10^{-3}$; $\mu \sim 10^{-2}$, $\rho \sim 1$; $\sigma \sim 10$; and $U_i \sim 10^{-2}$. It follows for the axis velocity $U_0 \equiv U(0)$ that $U_0 \sim U_i b/h \sim 1$. The pressure drop needed to drive the flow is found from [2] and [3] with $r = 0$:

$$\frac{\partial P}{\partial z} \sim \frac{U_0 \mu}{b^2} \sim \frac{1 \times 10^{-2}}{10^{-2}} \sim 1.$$

The time scale [38] is found to be $\sim 10^2$. The capillary instability tends to be a "slow" one: it will often have no time to show up, if the capillary is short so that the time it takes for the film fluid to pass through it is less than the time scale [38]. In our numerical example this means that the capillary is shorter than ~ 1 cm, however.

In real systems, the instability due to viscosity stratification (12) can prove to be more important. It can saturate by the same nonlinear mechanism (13). Finally, when the film is very thin, $h < 10^{-4}$ cm, the van der Waals molecular forces play the role of the destabilizing factor for the saturation mechanism (14).

The main results of the present work were presented also at the Fifth International Conference on Physico-Chemical Hydrodynamics (15).

APPENDIX

From the complete fluid equations (see, e.g., (3)) splitting the perturbed velocity $(\tilde{u}, \tilde{v}, \tilde{w})$ in the sum of the primary velocity $(U, 0, 0)$ and perturbations (u, v, w) as $(\tilde{u}, \tilde{v}, \tilde{w}) = (U, 0, 0) + (u, v, w)$, and the perturbed pressure $\tilde{p}$ as $\tilde{p} = P + p$, one easily finds the exact nonlinear equations for perturbations. The Navier-Stokes perturbation equations are

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial z} + u\frac{\partial u}{\partial z} + \frac{v}{r}\frac{\partial u}{\partial \phi} + w\frac{\partial U}{\partial r} + w\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u, \qquad [A1]$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial z} + u\frac{\partial v}{\partial z} + \frac{v}{r}\frac{\partial v}{\partial \phi} + w\frac{\partial v}{\partial r} + \frac{wv}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} + \nu\left(\nabla^2 v + \frac{2}{r}\frac{\partial w}{\partial \phi} - \frac{v}{r^2}\right), \qquad [A2]$$

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial z} + u\frac{\partial w}{\partial z} + \frac{v}{r}\frac{\partial w}{\partial \phi} + w\frac{\partial w}{\partial r} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 w - \frac{2}{r^2}\frac{\partial v}{\partial \phi} - \frac{w}{r^2}\right), \qquad [A3]$$

where ∇^2 is the three-dimensional Laplacian in cylindrical coordinates,

$$\nabla^2 \equiv \frac{\partial^2}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}. \qquad [A4]$$

The continuity equation is

$$\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial v}{\partial \phi} + \frac{1}{r}\frac{\partial}{\partial r}(rw) = 0. \qquad [A5]$$

We will denote the core quantities by the subscript "c" to distinguish them from the corresponding film quantities. We will not write down the core equations—they can be obtained from [A1–3] and [A5] formally by simply changing all symbols of perturbations u, v, w, p to u_c, v_c, w_c, p_c, correspondingly.

We turn now to the boundary conditions. On the capillary walls, the viscous film must satisfy the no-slip conditions,

$$u = v = w = 0, \quad (r = c). \qquad [A6]$$

On the fluid–fluid interface, the kinematic impermeability condition is

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial z} + u\frac{\partial \eta}{\partial z} + \frac{v}{r}\frac{\partial \eta}{\partial \phi} + w = 0. \qquad [A7]$$

Also, we must demand the continuity of velocities at the interface, as well as the equilibrium of forces there. The exact equations in cylindrical coordinates can be found in (7) [see also (3) and (4)]. We will not rewrite those rather lengthy equations here. Instead, we will give their approximate form justified by smallness of surface curvatures generated by [31] as it follows from [45], [41], [44], and [4]: the z–r stress condition

$$\frac{\partial w}{\partial z}+\frac{\partial u}{\partial r}=\frac{\partial w_c}{\partial z}+\frac{\partial u_c}{\partial r}, \quad (r=b); \qquad [A8]$$

the ϕ–r stress condition

$$\frac{1}{r}\frac{\partial w}{\partial \phi}+\frac{\partial v}{\partial r}-\frac{v}{r}=\frac{1}{r}\frac{\partial w_c}{\partial \phi}+\frac{\partial v_c}{\partial r}-\frac{v_c}{r}, \quad (r=b); \qquad [A9]$$

the r–r stress condition

$$p-2\mu\frac{\partial w}{\partial r}=p_c-2\mu\frac{\partial w_c}{\partial r} - \sigma\left(\frac{1}{b^2}\eta+\frac{\partial^2\eta}{\partial z^2}+\frac{1}{b^2}\frac{\partial^2\eta}{\partial\phi^2}\right), \quad (r=b); \qquad [A10]$$

and, finally, the velocity continuity

$$u=u_c, \quad v=v_c, \quad w=w_c, \quad (r=b). \qquad [A11]$$

Now, our derivations can be justified (in the way discussed earlier) in the parameter range defined by [1], [12], and [41] (see also [38]), as follows.

In our solution, the film pressure p is expressed in terms of η through [9], and then the film velocities are expressed in terms of η through [23], [25], and [30]. Since $\eta \sim \epsilon h$, see [40] and [41], one finds $(u/U_i) \sim (\sigma h^3\epsilon/\mu b^3 U_i) \sim \epsilon(b/TU_i) \sim \epsilon^2$ (thus justifying [20]). Analogously, $v \sim \epsilon^2 U_i$ and $w \sim \epsilon^2\kappa U_i$. Also, we have in the film $(1/r) \sim (1/b)$, $(\partial/\partial r) \sim (1/h)$, $(\partial/\partial z) \sim (1/b)$, and $(\partial/\partial t) \sim (1/T)$. Substituting the above estimates into [A1–3], one can easily see the smallness of the terms neglected in the transformation of those exact equations to [5]–[7]. For example, we estimate the ratio of the neglected term $w(\partial U/\partial r)$ in [A1] to the retained term $\nu(\partial^2 u/\partial r^2)$ to be small:

$$w\frac{\partial U}{\partial r}\bigg/\nu\frac{\partial^2 u}{\partial r^2}\sim\kappa\epsilon^2 U_i\frac{U_i}{h}\bigg/\nu\frac{\epsilon^2 U_i}{h^2} \sim\kappa\frac{U_i h}{\nu}\sim\kappa R\ll 1.$$

For the core, clearly, $(\partial/\partial r) \sim (1/b)$ instead of $1/h$ in the film, and also $U \sim U_i(b/h) = (U_i/\kappa)$ (see [3]). One estimates, taking into account the analog of [A5] for the core, that $w_c \sim u_c \sim v_c$, and, from the continuity of velocities at the interface [11], $w_c \sim v_c \sim u_c \sim u \sim \epsilon^3 U_i$. Then it is easy to justify that the core analog of [A1–3] can be approximated as

$$U\frac{\partial u_c}{\partial z}+w_c\frac{\partial U}{\partial r}=-\frac{1}{\rho}\frac{\partial p_c}{\partial z},$$

$$U\frac{\partial v_c}{\partial z}=-\frac{1}{\rho r}\frac{\partial p_c}{\partial \phi},$$

$$U\frac{\partial w_c}{\partial z}=-\frac{1}{\rho}\frac{\partial p_c}{\partial r}$$

which leads to estimating the pressure as

$$p_c \sim \rho\frac{U_i^2\epsilon^2}{\kappa}.$$

In the film, from [5],

$$p \sim b\nu\rho\frac{U_i\epsilon^2}{h^2}. \qquad [A12]$$

Then

$$p_c/p \sim \frac{\rho U_i^2\epsilon^2}{\kappa}\,\frac{h^2}{b\nu\rho U_i\epsilon^2}\sim\frac{U_i h}{\nu}=R\ll 1.$$

Hence, p_c can be neglected in [A10]. Neglecting as well the viscous terms in [A10] can be justified using [A12]; for example,

$$\mu\frac{\partial w_c}{\partial r}\bigg/p \sim \frac{\mu\epsilon^2 U_i h^2}{b b\mu U_i\epsilon^2}\sim\kappa^2\ll 1.$$

Thus, reducing [A10] to [8] is justified.

Finally, it is easy to see that all the terms in [A8] other than $\partial u_i/\partial r$ are indeed much smaller than $\epsilon^2 U_i/h$, the order of $\partial u_i/\partial r$ in the film. This justifies [24]. Considering [A9] analogously justifies the equation $(\partial v/\partial r) = 0$ $(r = b)$ which was used to arrive at [25].

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