Dragging of a Liquid by a Moving Plate

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One of the methods of depositing a thin layer of liquid upon a surface of a solid, which is wetted by it, consists of pulling out an infinite film with constant speed v_0.

This method is applied, in particular, in the cinefilm manufacture with the purpose of depositing a uniform layer of photosensitive emulsion upon the cinefilm base.

From the trough containing the dissolved photosensitive emulsion an infinite film, wetted by the solution, is pulled out.

After the solvent has evaporated, a uniform layer of emulsion is left deposited upon the surface of the film base.

The problem of determining the thickness of the dragged layer as a function of the speed of the motion of the film and of parameters characteristic of the properties of the fluid (its viscosity η, its surface tension σ and its density ρ) is of essential interest for practice.

Numerous attempts at evaluating the thickness of dragged layer of fluid found in literature contain some incorrect assumptions in the very basis of the method of computation, thus leading to erroneous formulae for the value of this thickness.

In the present paper the thickness of the layer and the quantity of fluid carried along when pulling an infinite plate out of a vessel, which is sufficiently large to permit the neglecting of the effect of its walls and of the edges of the plate, is evaluated.

Let us choose a set of coordinates in such a way as to make the plate — a plane $x=0$ and the surface of the liquid undisturbed by the capillary meniscus (*i. e.* sufficiently far from the plate) — the plane $z=0$, with the z axis directed upward along the plate.

First of all, let us consider the case of low velocity of motion of the plate (it will be stated below which velocities can still be considered low).

In this case all the surface of the liquid may be separated into two independent regions: the region of the surface situated high above the meniscus and directly dragged by the plate, where the surface of liquid may be taken to be nearly parallel to the plate surface, and the region of the meniscus of liquid, which will be slightly deformed by the motion of the plate, hence the shape of the surface will nearly coincide with the shape of static meniscus.

Below we shall write down the solutions of hydrodynamical equations in both independent regions and then connect both of the solutions found.

Let us denote by h the thickness of a layer of liquid, when measured from the plate. We shall look for $h=h(z)$ in both of the independent regions. First of all, let us write down the equations for the thickness of the liquid film carried along by the moving plate, *i. e.* the equations for h in the first region.

Since the surface of the liquid in the firsf region is nearly flat, if is clear that the motion of the moving liquid in this region will also be nearly flat. In other words, the main component of the motion of the liquid in the first region will be the flowing down of it nearly parallel to the plate surface.

This peculiarity may be used in order to simplify the equations of hydrodynamics suitably for this case.

In fact, it is evident that the only component of the velocity of fluid which plays an essential part is the vertical (along the z axis) component u_z.

It is evident as well that the gradients of the velocity along the direction normal to the plate $\frac{\partial u_z}{\partial x}$ are large as compared with the gradients of velocity along the plate $\frac{\partial u_z}{\partial z}$.

Therefore the motion of a liquid carried along with the plate is described by the equations of the Prandtl boundary layer, which in the stationary case have the form:

$$\nu \frac{d^2u}{dz^2} = \frac{1}{\varrho}\frac{\partial p}{\partial z} + g, \tag{1}$$

$$\frac{\partial p}{\partial x} = 0. \tag{2}$$

Here, for the sake of brevity, the z component of the velocity u_z is designated simply by u. Other symbols have their usual meaning.

As the boundary conditions for equations (1) and (2) the following conditions will serve: on the surface of the plate, no slip between the liquid and the plate occurs, hence

$$u = v_0 \quad \text{at} \quad x = 0. \tag{3}$$

Here v_0 is the velocity of motion of the plate.

On the free surface of the liquid, at $x = h(z)$, where h is the thickness of the liquid layer, the pressure inside the liquid must be equal to the capillary pressure p_σ and, moreover, the tangent stress must be absent, so that:

$$\left.\begin{aligned} p &= p_\sigma, \\ \eta \frac{\partial u}{\partial x} &= 0, \end{aligned}\right\} \quad x = h. \tag{4}$$

The capillary pressure is known to equal

$$p_\sigma = \frac{\sigma}{R},$$

σ being the surface tension, R — the radius of curvature of the surface.

Substituting the well-known expression for the radius of curvature R, we get:

$$p_\sigma = -\sigma \frac{\dfrac{d^2h}{dz^2}}{\left[1 + \left(\dfrac{dh}{dz}\right)^2\right]^{3/2}}. \tag{5}$$

As long as the thickness of the layer of the liquid carried along is very small, it is evident that the curvature of the surface of the liquid in the vicinity of the plate also will be very small. Therefore, the square of dh/dz in the denominator of equation (5) may be neglected, and the following equation for the capillary pressure in the region of the liquid carried along may be finally written down:

$$p_c \approx -\sigma \frac{d^2h}{dz^2}. \tag{5'}$$

Therefore, the first of the boundary conditions (4) may be rewritten as follows:

$$p = -\sigma \frac{d^2h}{dz^2} \quad \text{at} \quad z = h.$$

However, it follows from equation (2) that the pressure is constant along the thickness of the liquid layer carried along. Thus, not only on the free surface, but throughout inside the liquid pressure is also given by

$$p = -\sigma \frac{d^2h}{dz^2}. \tag{6}$$

The solution of equation (1) satisfying the boundary conditions (3) and (4) is:

$$u = v_0 + \left(\frac{1}{\eta}\frac{dp}{dz} + \frac{\rho g}{\eta}\right)\left(\frac{x^2}{2} - hx\right) =$$
$$= v_0 + \left(\frac{\rho g}{\eta} - \frac{\sigma}{\eta}\frac{d^3h}{dz^3}\right)\left(\frac{x^2}{2} - hx\right) \tag{7}$$

where the value of p is substituted from equation (6).

Let us finally make use of the continuity equation, in order to connect the thickness of the liquid layer with the flow of the fluid carried along by the plate.

For steady motion of the fluid, keeping in mind the incompressibility of fluid, we may, evidently, write down the continuity equation in the form:

$$j = \int_0^h u\, dh = \text{const},$$

where j is the flux of the fluid per unit of width of the plate. Substituting the value of u from equation (7), we obtain:

$$v_0 h - \left(\rho g - \sigma \frac{d^3h}{dz^3}\right)\frac{h^3}{3\eta} = j. \tag{8}$$

Equation (8) defines the thickness of liquid layer $h(z)$ far from the surface of the fluid (in the first region [1]).

In the second region, near the fluid surface, the thickness $h(z)$, as has been said above, is determined by the equation for the static meniscus.

Let us rewrite equation (8) in a more suitable form:

$$\frac{d^3h}{dz^3} = \frac{3\eta}{\sigma} \frac{(j - v_0 h)}{h^3} - \frac{\rho g}{\sigma}. \tag{8'}$$

Let us introduce a new dimensionless coordinate λ determined by the equation

$$\lambda = \left(\frac{\sigma}{3\eta}\right)^{1/3} \frac{j}{v_0^{4/3}} z, \tag{9}$$

and a dimensionless expression for the thickness of the liquid layer

$$\mu(\lambda) = \frac{v_0 h(z)}{j}. \tag{10}$$

Then, introducing λ and $\mu(\lambda)$ into equation (8'), we find:

$$\frac{d^3\mu}{d\lambda^3} - \frac{1-\mu}{\mu^3} - \frac{\rho g j^2}{3\eta v_0^3} = 0. \tag{11}$$

The order of magnitude of the last term in equation (11) which contains all the dimensional quantities involved in the problem, is determined by the kind of dependence of the flux j on the velocity of the plate v_0. If the flux j is proportional to v_0 in the power higher than $^2/_3$, then the last term in equation (11) will be simply proportional to v_0, and for sufficiently small values of velocity of elevation v_0 will be small as compared with unity.

We assume the flux j to depend on v_0 in the way mentioned above. Then the last term in equation (11) is in fact small as compared with both the first ones and may therefore be neglected. It will be shown later that this assumption turns out to hold within some regions of velocities of elevation of the plate. Thus, the region of validity of solutions of equation (11) obtained on the basis of this assumption will be defined.

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[1] One of us (B. Levich) has been kindly informed by B. V. Derjaguin that he was first to obtain this equation. However, B. V. Derjaguin, has failed to derive from it any particular conclusions concerning the thickness of the film carried along.

Neglecting the last term in equation (11), we get finally:

$$\frac{d^3\mu}{d\lambda^3}=\frac{1-\mu}{\mu^3}. \tag{12}$$

The following conditions serve as the boundary conditions for equation (12): on the upper boundary of the region, for very large values of z (far from the surface of the liquid) the thickness of the liquid layer h must tend to a constant limit h_0, which evidently equals:

$$h_0 \to \frac{f}{v_0} \qquad \text{for } z \to \infty.$$

The derivatives dh/dz and d^2h/dz^2 must in this case tend to zero.

Therefore, in terms of dimensionless quantities λ and μ, the boundary conditions may on the upper boundary of the region be written down in the form:

$$\left.\begin{array}{l} \mu \to 1 \\ \dfrac{d\mu}{d\lambda} \to 0 \\ \dfrac{d^2\mu}{d\lambda^2} \to 0 \end{array}\right\} \quad \text{for } h \to \infty. \tag{13}$$

In order to determine the boundary conditions on the lower boundary of the region, at small values of z (near the meniscus), let us turn to the equations for the static meniscus, which determines the shape of the surface of the layer in the second region.

The equations for the static meniscus have, as is known, the following form:

$$\frac{\dfrac{d^2h}{dz^2}}{\left[1+\left(\dfrac{dh}{dz}\right)^2\right]^{3/2}}=\frac{\rho g z}{\sigma}. \tag{14}$$

Integrating equation (14), we find

$$\frac{\dfrac{dh}{dz}}{\left[1+\left(\dfrac{dh}{dz}\right)^2\right]^{1/2}}=\frac{\rho g z^2}{2\sigma}+c.$$

The constant of integration may be determined from the boundary conditions far away from the plate. In this case precisely,

$z \to 0$ and $dh/dz \to 0$, *i. e.* the surface of the fluid is horizontal. Therefore $c = -1$ and

$$\frac{\frac{dh}{dz}}{\left[1 + \left(\frac{dh}{dz}\right)^2\right]^{1/2}} = \frac{\rho g z^2}{2\sigma} - 1. \tag{15}$$

The quantity $\left(\frac{\rho g}{\sigma}\right)^{1/2}$ entering the equations (14) and (15) is the Laplace capillary constant, having dimensionality of the length. The capillary constant is the characteristic length of the problem of the static meniscus. If we shall designate it by a, then equation (14) and (15) may be written down in the form:

$$\frac{\frac{d^2h}{dz^2}}{\left[1 + \left(\frac{dh}{dz}\right)^2\right]^{3/2}} = \frac{z}{a^2}, \tag{14'}$$

$$\frac{\frac{dh}{dz}}{\left[1 + \left(\frac{dh}{dz}\right)^2\right]^{1/2}} = \frac{z^2}{2a^2} - 1. \tag{15'}$$

Equations (14) and (15) determine the thickness of the layer of liquid $h(z)$ in the second region close to the meniscus.

At small values of $h(z)$, small as compared with the capillary constant, and large values of z, the solution of the equality (14) must evidently go over into the solution of the equation (12) for the thickness of the film carried along. We must, therefore, chain together the solution of equation (12) with that of equation (11). The conditions of chaining together of both the equations will at the same time serve at the sought for boundary conditions of equation (11) at the lower boundary of the first region.

The conditions of chaining together of both solutions shall be obtained from equation (15') with the aid of the transition to the limit of the small thicknesses $h \to 0$.

It is clear that with h tending to zero the quantity dh/dz in the formula (15') will tend to zero as well. Since the surface of the liquid wetting the film in the vicinity of the film itself would be almost vertical, we find at the same time from equation (15) that to the thickness, tending to zero and to the almost vertical surface of the liquid, the finite altitude, tending to the limit value

$$z \longrightarrow 2a \tag{16}$$

corresponds.

Next, with the aid of equation (14) we find at the same time that the second derivative of the thickness d^2h/dz^2 in the static solution tends to the limit

$$\frac{d^2h}{dz^2} \longrightarrow \frac{2}{a}\,. \tag{17}$$

Going over to the dimensionless coordinate λ and to the thickness μ, we find, with the aid of equations (19), (10), (16), (17):

$$\mu \longrightarrow 0 \quad \begin{cases} \lambda \longrightarrow \left(\dfrac{\sigma}{3\eta}\right)^{1/3} \dfrac{2ja}{v_0^{4/3}}\,, \\ \dfrac{d^2\mu}{d\lambda^2} \longrightarrow \dfrac{\sqrt{2}\, a\, \sigma^{2/3} j}{v_0^{5/3}\,(3\eta)^{2/3}}\,. \end{cases} \tag{18}$$

Thus, we see that the boundary with the first region, $\frac{d^2\mu}{d\lambda^2}$, tends to a constant limit, determined by equation (18).

Let us turn now to the boundary conditions of equation (11) at the lower boundary of the region. Here the thickness of the film of the liquid carried along will be large as compared with the limit thickness $\mu = 1$ at the upper boundary of the region. In other words, the valuea of μ tending to infinity correspond to the lower boundary of the region. Therefore, we must find the boundary conditions of equation (11) with $\mu \longrightarrow \infty$. Thus, the boundaries of both independent regions overlap. $\mu \longrightarrow \infty$ corresponds to the lower boundary of the first region, $\mu \longrightarrow 0$ — to the upper boundary of the second region.

We shall require, as a condition for chaining together the solutions of both regions, the continuity of the second derivative $\frac{d^2\mu}{d\lambda^2}$.

From the point of view of geometry, the continuity of $\frac{d^2\mu}{d\lambda^2}$ expresses the continuity of the curvature of the surface in the region of small curvatures. If we shall designate by α the limit $\lim \left(\frac{d^2\mu}{d\lambda^2}\right)_{\mu\to\infty}$, where μ is the solution of equation (11), then with the aid of equation (18) the condition of chaining together the solutions for both regions may be written in the form:

$$\left(\frac{d^2\mu}{d\lambda^2}\right)_{\mu\to 0} = \frac{\sqrt{2}\, a\, \sigma^{2/3} j}{v_0^{5/3} (3\eta)^{2/3}} = \alpha. \tag{19}$$

Inasmuch as no dimensional quantities enter equation (11), α is a pure number. The quantity α may be found by means of the numerical integration of equation (11), which will be performed below.

Equation (19) gives us

$$j = \frac{\alpha}{\sqrt{2}} \cdot \frac{v_0^{5/3} (3\eta)^{2/3}}{\sigma^{2/3} a}. \tag{20}$$

The condition (20) will be discussed later on. In order to perform the numerical integration (11), it is necessary to investigate in more detail the character of the tending of the derivatives $d\mu/d\lambda$ and $d^2\mu/d\lambda^2$ to zero with λ increasing infinitely, *i. e.* at the upper boundary of the region [see equation (13)]. It may be established from the behaviour of the asymptotic solutions of equation (11).

Namely, for sufficiently large values of λ, the thickness μ may, evidently, be represented in the form:

$$\mu(\lambda) = 1 + \mu_1(\lambda) \tag{21}$$

with $\mu_1(\lambda) \ll 1$.

Then substituting the value $\mu(\lambda)$ from equation (21) into equation (11), we find, after neglecting the squares of small quantities, the linear equation for μ_1:

$$\frac{d^3\mu_1}{d\lambda^3} = -\mu_1. \tag{22}$$

As the boundary conditions for equation (22) serves:

$$\mu_1 \to 0 \quad \text{for} \quad \lambda \to \infty.$$

As a particular solution of equation (22), satisfying this boundary condition, may serve:

$$\mu_1 = \text{const} \cdot e^{-\lambda}.$$

It is evident at the same time that

$$\frac{d\mu_1}{d\lambda} = -\operatorname{const} \cdot e^{-\lambda} = -\mu_1,$$

$$\frac{d^2\mu_1}{d\lambda^2} = \operatorname{const} \cdot e^{-\lambda} = \mu_1.$$

Thus, at large values of λ we have the following asymptotic equations for μ and for its derivatives:

$$\mu = 1 + \operatorname{const} \cdot e^{-\lambda}, \tag{23}$$

$$\frac{d\mu}{d\lambda} = 1 - \mu, \tag{23'}$$

$$\frac{d^2\mu}{d\lambda^2} = \mu - 1. \tag{23''}$$

The values of the constant figuring in equation (23) turn out to be unessential for our purpose. For the direct performance of the numerical integration of equation (11), with boundary conditions(23), (23′) and (23″) kept in mind, it would be convenient to lower the order of the equation, choosing μ as the new variable quantity, and $(d\mu/d\lambda)^2$ as the new unknown function. If we shall designate $(d\mu/d\lambda)^2$ by ξ, then, after simple transformations, we find the following equation for ξ:

$$\frac{d^2\xi}{d\mu^2} = \frac{2(1-\mu)}{\mu^3 \sqrt{\xi}}. \tag{24}$$

The boundary conditions (23) — (23″) may be now written down as

$$\left.\begin{aligned} &\xi \to (1-\mu)^2, \\ &\frac{d\xi}{d\mu} \to 2(\mu - 1) \end{aligned}\right\} \tag{25}$$

for $\mu \to 1$.

We are directly interested in the quantity α equal in the new designations to

$$\alpha = \lim \left(\frac{d^2\mu}{d\lambda^2}\right)_{\mu\to\infty} = \frac{1}{2} \lim \left(\frac{d\xi}{d\mu}\right)_{\mu\to\infty}.$$

The numerical analysis of equation (18) with boundary condition (27) gives for α the value

$$\alpha = 0.63 \ldots$$

Substituting this value of α into equation (20) for the flux of the fluid, we get finally

$$j = 2.29 \cdot \frac{\eta^{2/3} v_0^{5/3}}{\sigma^{1/6} (\rho g)^{1/2}} . \tag{20'}$$

We see thus that the conditions of chaining together of the solutions in both regions allow to express the flux j by the characteristic quantities of our problem. It follows from equation (20) that the quantity of the fluid carried along by a slowly moving plate is directly proportional to the velocity of the movement v_0 to the power $^5/_3$, and to the viscosity of the fluid to the power $^2/_3$, and is inversely proportional to the tension to the power $^1/_6$ (*i. e.* it shows a very slight dependence on the surface tension).

The limit thickness of the layer of the liquid carried along by the plate far away from the meniscus of the fluid, *i. e.* at $\mu = 1$, would be [*cf.* (10)]:

$$h_0 = 2.29 \frac{(v_0 \eta)^{2/3}}{\sigma^{1/2} \sqrt{\rho g}} . \tag{26}$$

We see, therefore, that the limit thickness of the layer of liquid carried along is proportional to the velocity of the elevation of the plate and to its viscosity to the power $^2/_3$, and shows a rather slight dependence on the surface tension, being inversely proportional to it to the power $^1/_6$.

Let us ascertain now the conditions of the applicability of the formulae received [equations (20) and (21)] for the consumption and for the thickness of the layer carried along.

In going over from (15) to (16), we omit the last term of equation (15), on the assumption that the quantity $\rho g j^2/3\eta v_0^3$ is small as compared with unity.

Substituting for j its value obtained from formula (22), we find that this quantity is really small as compared with unity, and our calculation is legitimate if the following inequality is fulfilled:

$$\left(\frac{\eta v_0}{\sigma}\right)^{1/3} \ll 1, \quad i.\ e. \quad v_0 \ll \frac{\sigma}{\eta} , \tag{27}$$

i. e. at suificiently small values of the velocity of the plate.

The expressions obtained for j and h seem to be in good accordance with the experiment. Indeed, the experiment shows that the thickness of the layer carried along is proportional at small velocities to v_0 to the power 0.6, which agrees with the power index obtained by us.

In the opposite limiting case, when the velocity v_0 turns out to be greater than σ/η, the given calculation becomes inapplicable. Namely, the supposition that all the surface of the fluid may be split into two independent regions, which has led to all the expressions written above, does not hold here any longer. It is impossible to obtain in this case the exact expressions for j and h. However, on the basis of dimensionality considerations, the general character of the dependence of these quantities on the fundamental parameters v_0, η, ρ and σ at sufficiently large velocity may be pointed out.

Namely, it is clear that at high velocities the consumption j and the thickness h of the layer carried along should not depend on the surface tension. From the viewpoint of physics, this may be seen from the fact that at sufficiently high velocities of the plate, the shape of the entire surface of the fluid would be determined by the character of the process oi carrying along, and not by the static properties of the surface of the fluid. Therefore, at high vefocities of the plate, the thickness of the layer carried along must depend only on the quantities v_0, η, ρ and g. The only quantity of the dimensionality of length, which may be obtained from these quantities, is the quantity $(\eta v/\rho g)^{1/2}$.

Therefore, at sufficiently high velocities of the elevation of the plate, the thickness of the layer of liquid carried along must have the form:

$$h \approx A\left(\frac{\eta v_0}{\rho g}\right) \tag{28}$$

and the consumption

$$j \approx A\left(\frac{\eta}{\rho g}\right)^{1/2} v_0^{2/3}.$$

The numerical value of the constant A may be found only by means of experiment.

Finally, in the intermediate region of velocities, it may be seen out of dimensionality considerations that the thickness of the layer carried along must have the following form:

$$h = \left(\frac{\eta v_0}{\rho g}\right)^{1/2} f\left(\frac{v_0 \eta}{\sigma}\right) \tag{29}$$

where $f(v_0\eta/\sigma)$ is some function of the dimensionless parameter $v_0\eta/\sigma$, the form of which must be found by experiment.

In both the limit cases of large and small values of $v_0\eta/\sigma$ the function $f(v_0\eta/\sigma)$ tends accordingly to unity and to $(v_0\eta/\sigma)^{1/6}$, respectively, so that equation (29) goes over into equations (28) or (26).

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