CALCULATION OF THE DISTRIBUTION OF PROTECTIVE POTENTIAL AND CURRENT ON A FLAT METAL SURFACE

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It is known that one of the most effective methods of combatting corrosion of various structures and devices is the method of electrochemical protection. At the present time variations of this method—protector and cathode shielding—are becoming more and more widely used to prevent corrosion of such different structures as sea ships and stockades, various conduits, reservoirs, etc. In certain instances the method of electrochemical protection is essentially the only effective method of reducing corrosion of metal structures.

All this indicates the great importance of working out scientifically sound methods of planning protector and cathode systems.

Up to the present time the focus of attention in the solution of this problem has been on the electrochemical aspects which have been examined in detail in a very large number of works. Much less studied have been the physical questions of electrochemical corrosion and protection of metals—questions of the distribution of protective potential and current. In rigorous formulation these questions have been studied in comparatively few works, among the most notable of which must be mentioned the works of C. Wagner [1-3], A. N. Frumkin [4], V. G. Levich [4-7], J. Waber [8-9], C. Plumpton [11-13], and others. In addition to these, a number of works have been published on the topic of distribution of protective potential and current which are based mainly on approximation of approximate experimental relations or on the study of exceedingly simple computational models. Undoubtedly the solution of such problems would be much more successful if the well-developed theory of the electromagnetic field were employed.

Actually, the computation of the distribution of potential and current in any conducting medium (including a corrosion medium) is a subject in the theory of the static electric field similar to that studied in theoretical electrotechnics. However, this calculation can be performed only with given boundary conditions for the potential on the shielded surface, the formulation of which is possible only on the basis of a study of the electrochemical mechanism of corrosion and protection of the metal. Thus both sides of the question—the physical and the electrochemical—are inextricably related and cannot be separated one from the other.

The present article is devoted to the calculation of the law of distribution of protective potential and current on flat sections of a shielded metal surface, i.e., on the simplest, but at the same time, most typical shape of surface.*

Let us first consider the case where a flat section of metal surface is shielded by a protector located at a distance h from this surface (Fig. 1). We shall assume the specific polarization of the shielded metal to be constant and equal to b, while the polarization of the protector as well as its dimensions (as compared with h) we shall neglect. Then, as shown in [14], the problem of finding the protective potential (and then the current density) on the surface of the metal reduces to integrating the Poisson equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)+\frac{\partial^2 U}{\partial z^2}=-f(r, z), \qquad (1)$$

*To be specific, we shall subsequently study protector shielding, although all the conclusions may be applied equally well to the case of cathode shielding.

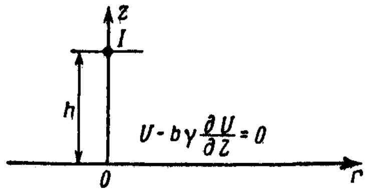

Fig. 1. Computation of the distribution of protective potential and current of a single protector situated near a plane, linearly polarized metallic surface.

where

$$f(r, z)=\begin{cases}\dfrac{I}{\gamma} & \text{if} \quad r=0;\ z=h;\\ 0 - \text{at all other points;}\end{cases}$$

γ is the specific electric conductivity of the corrosion medium; and the boundary condition is

$$U-b\gamma\frac{\partial U}{\partial z}=0. \tag{2}$$

We shall seek the solution by introducing a new function φ related to the potential U of interest to us by the relation:

$$U=\varphi+\frac{I}{4\pi\gamma}[r^2+(z-h)^2]^{-1/2}. \tag{3}$$

Inasmuch as the second member of the right-hand side of this equation determines the potential of a point source in free space, it is, as in U also, a harmonic function at all points of space except the points z = h; r = 0. From this it follows that φ is a harmonic function at all points of the half-surface $z \geq 0$ including the point r = 0; z = h, i.e., finding φ is reduced to integrating the Laplace equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right)+\frac{\partial^2\varphi}{\partial z^2}=0 \tag{4}$$

with the boundary condition

$$\varphi-b\gamma\frac{\partial\varphi}{\partial z}\bigg|_{z=0}=-\frac{I}{4\pi\gamma}[(r^2+h^2)^{-1/2}-b\gamma h(r^2+h^2)^{-3/2}]. \tag{5}$$

We shall seek the solution of this problem by means of an integral Hankel transform, i.e., we shall represent the expression for φ in the form:

$$\varphi=\int_0^\infty A(p)J_0(pr)e^{-pz}\,dp, \tag{6}$$

where $J_0(pr)$ is the Bessel function of first kind of order zero.

To determine the unknown factor A(p) which enters here, we substitute this expression into the boundary condition (5)

$$\int_0^\infty A(p)(1+b\gamma p)J_0(pr)\,dp=-\frac{I}{4\pi\gamma}[(r^2+h^2)^{-1/2}-b\gamma h(r^2+h^2)^{-3/2}] \tag{7}$$

and then make use of the known decompositions [15]:

$$(r^2+h^2)^{-1/2}=\int_0^\infty e^{-hp}J_0(pr)\,dp, \qquad \frac{h}{(r^2+h^2)^{3/2}}=\int_0^\infty pe^{-hp}J_0(pr)\,dp.$$

Equation (7) can then be written in the form

$$\int_0^\infty A(p)(1+b\gamma p)J_0(pr)\,dp = -\frac{I}{4\pi\gamma}\int_0^\infty (1-b\gamma p)e^{-ph}J_0(pr)\,dp,$$

whence we find that

$$A(p) = \frac{I}{4\pi\gamma}\,\frac{b\gamma p-1}{b\gamma p+1}e^{-ph}.$$

Putting the value of A(p) obtained into the expression for φ and then returning (by means of equation (3)) to the potential U, we find:

$$U = \frac{I}{4\pi\gamma}\left[\frac{1}{\sqrt{r^2+(z-h)^2}} + \int_0^\infty \frac{b\gamma p-1}{b\gamma p+1}e^{-p(z+h)}J_0(pr)\,dp.\right.$$

Putting z = 0 in this formula, we obtain (after rearranging) the following expression for the protective potential on the metal surface:

$$U|_{z=0} = \frac{I\cdot b}{2\pi}\int_0^\infty \frac{p}{b\gamma p+1}e^{-ph}J_0(pr)\,dp. \tag{8}$$

This formula makes it possible also to find the expression for the density of protective current which, as is evident from boundary condition (2), takes the following form:

$$j|_{z=0} = \frac{I}{2\pi}\int_0^\infty \frac{p}{b\gamma p+1}e^{-ph}J_0(pr)\,dp. \tag{9}$$

Precise expression for the integrals entering formulas (8) and (9) have not yet been found; for their approximate evaluation we replace the ratio $\dfrac{p}{1+b\gamma p}$ entering there by the function $\dfrac{1}{b\gamma}(1-e^{-b\gamma p})$, the value of which, as one can verify, is extremely close to the value of the function it replaces both for large and small values of the argument p. The approximate expressions for the protective potential and current density then assume the following simple form:

$$U|_{z=0} \simeq \frac{I}{2\pi\gamma}\left[\frac{1}{\sqrt{r^2+h^2}} - \frac{1}{\sqrt{r^2+(h+b\gamma)^2}}\right] \tag{10}$$

$$j|_{z=0} \simeq -\frac{I}{2\pi b\gamma}\left[\frac{1}{\sqrt{r^2+h^2}} - \frac{1}{\sqrt{r^2+(h+b\gamma)^2}}\right]. \tag{11}$$

It is expedient to use these expression for values of r > 0. When r = 0 the integrals in formulas (8) and (9) simplify considerably and after easy maniuplations reduce to the following precise expressions for the maximum values of potential and current density on the shielded surface:

$$\left.\begin{aligned} U|_{\substack{z=0\\r=0}} &= \frac{I}{4\pi\gamma}\left[\frac{1}{h}+\frac{1}{b\gamma}e^{h/b\gamma}Ei\left(-\frac{h}{b\gamma}\right)\right] \\ j|_{\substack{z=0\\r=0}} &= -\frac{I}{4\pi b\gamma}\left[\frac{1}{h}+\frac{1}{b\gamma}e^{h/b\gamma}Ei\left(-\frac{h}{b\gamma}\right)\right] \end{aligned}\right\}, \tag{12}$$

where Ei(x) is the exponential-integral function, the values of which are tabulated.

We shall make use of the results presented to determine the dimensions of the zone of protective action and the required current of the protector at given maximal and minimal values of the density of protective current. For this purpose, putting $j|_{\substack{z=0\\r=0}} = j_{\max}{}^0$ in the second of the equations (12) and solving this equation for the protector current, we obtain:

$$I = j^0_{\max} \frac{4\pi h b^2 \gamma^2}{b\gamma + h e^{h/b\gamma} Ei\left(-\frac{h}{b\gamma}\right)}. \tag{13}$$

Now taking $j_{z=0} = j^0_{\min}$ in formula (11), assuming $h/r_{pro} < 1$, and solving the equation so obtained for r_{pro}, we find the expression for the radius of protective action of the protector:

$$r_{pro} \simeq \left[\frac{I(2h + b\gamma)}{4\pi j^0_{\min}}\right]^{1/3}. \tag{14}$$

The general solution obtained for the distribution of protective potential and current taking polarization into account ($b \neq 0$) makes it easy to go over the case of no polarization ($b = 0$). We hereby obtain the well-known [16] expression for the primary distribution of protective current density:

$$j|_{z=0} = -\frac{I \cdot h}{2\pi (r^2 + h^2)^{3/2}}. \tag{15}$$

Let us now consider the system represented in Fig. 2 which refers to the case of protecting a flat section of metal surface with a system of protectors placed at the same distances from one another, these distances being commensurate with the distance to the protected surface. In order to determine the distribution of protective potential and current along the x-axis, let us assume that the length of the protectors in the direction perpendicular to this axis is infinitely large; this makes it possible to consider the problem in which we are interested in a two-dimensional formulation. The polarizability of the protected surface we shall assume, as before, to be constant and equal to b, and we shall neglect the polarizability of the protector and its cross sectional dimensions.

In view of the periodicity of the system along the x-axis, it is sufficient to consider the distribution of protective potential and current only within the limits of the section $-a \leq x \leq a$; in this case we arrive at the computational model shown in Fig. 3. We shall consider the current I flowing off from unit length of each protector to be known. Then the potential in the space enclosed by the protected surface will satisfy the two-dimensional Poisson equation:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = f(x, y), \tag{16}$$

where

$$f(x, y) = \begin{cases} \frac{I}{\gamma} & \text{if} \quad x = 0 \text{ and } y = h \\ 0 - \text{at all other points.} \end{cases}$$

For convenience of solution we shall first assume that

$$f(x, y) = \begin{cases} \frac{I}{\gamma \varepsilon^2} & \text{if} \quad |x| < \varepsilon \text{ and } |y - h| < \varepsilon \\ 0 - \text{at all other points.} \end{cases}$$

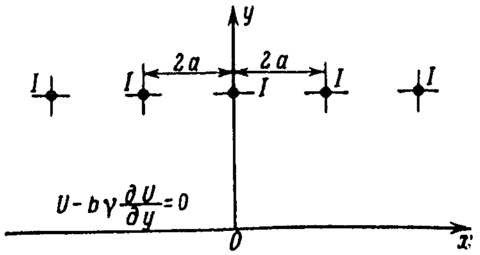

Fig. 2. For computation of the distribution of protective potential and current of a system of protectors situated near a flat linearly polarized metal surface.

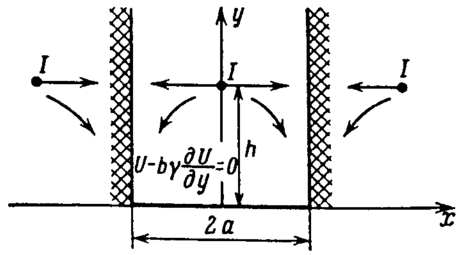

Fig. 3. Auxiliary computational model for determining the protective potential and current of the system of protectors.

We shall subsequently free ourselves of this approximation by a limiting process.

The boundary conditions which the potential must satisfy have the following form:

$$\left.\begin{aligned} &U - b\gamma\frac{\partial U}{\partial y} = 0 \quad \text{if} \quad |x| < a; \quad y = 0 \\ &\frac{\partial U}{\partial x} = 0 \quad \text{if} \quad |x| = a; \quad y > 0 \end{aligned}\right\} \tag{17}$$

We shall solve this problem by the method of G. A. Grinberg [17]. According to this method, we seek the solution in the form:

$$U = \sum_{(n)} \frac{\overline{U}}{N_n} X_n, \tag{18}$$

where X_n is an eigenfunction of the corresponding problem with zero boundary conditions, $N_n = \int_{-a}^{a} X_n^2\, dx$ is the norm of the eigenfunction, and $\overline{U} = \int_{-a}^{a} U X_n dx$ is the result of the finite integral transform of the function U with kernel X_n.

To find X_n we employ the usual method of separation of variables, and as a result we find:

$$X_n = \cos\frac{n\pi}{a}x, \text{ where } n = 0,\ 1,\ 2,\ \ldots$$

Then, making use of the expression given for N_n, we obtain

$$N_n = a.$$

Thus, to determine U from formula (18), it remains only to find the expression for $\overline{U}$. For this purpose we perform a finite integral transform (with kernel X_n) on equation (16) and as a result arrive at the equation:

$$\frac{d^2\overline{U}}{\partial y^2} - \frac{n^2\pi^2}{a^2}\overline{U} = \begin{cases} \dfrac{Ia}{2\gamma\varepsilon^2 hn\pi} & \text{if} \quad |y-h| < \varepsilon, \\ 0 & \text{if} \quad |y-h| > \varepsilon. \end{cases}$$

As is known, the general solution of this equation has the following form:

$$\bar{U} = Ae^{(n\pi/a)y} + Be^{-(n\pi/a)y} + \frac{a}{n\pi}\int_0^y \bar{f}(\xi)\,\mathrm{sh}\,\frac{n\pi}{a}(y-\xi)\,d\xi,$$

where

$$\bar{f}(\xi) = \begin{cases} \dfrac{aI\sin\dfrac{n\pi}{a}\varepsilon}{2\gamma\varepsilon^2 n\pi} & \text{if} \quad |y-h| < \varepsilon; \\ 0 & \text{if} \quad |y-h| > \varepsilon. \end{cases}$$

Performing the integration and then passing to the limit $\varepsilon \to 0$, we find:

$$\bar{U} = Ae^{(n\pi/a)y} + Be^{(n\pi/a)y} + f_1(y), \tag{19}$$

where

$$f_1(y) = \begin{cases} \dfrac{I\,\mathrm{sh}\,\dfrac{n\pi}{a}(y-h)}{2\gamma} & \text{if} \quad y > h; \\ 0 & \text{if} \quad y < h. \end{cases}$$

Determining the unknown coefficients A and B from the boundary conditions (17) and then inserting them in (19), we obtain finally the following expression for $\bar{U}$:

$$\bar{U} = \frac{I}{4\gamma}\left[e^{-(n\pi/a)(h-y)} + \frac{n\pi\dfrac{b\gamma}{a} - 1}{n\pi\dfrac{b\gamma}{a} + 1}e^{-(n\pi/a)(h+y)}\right]\frac{a}{n\pi}. \tag{20}$$

Then, inserting this value of $\bar{U}$ in formula (18) and letting $y \to 0$, we arrive at the following expression for the protective potential.

$$U|_{y=0} = \frac{Ib}{2a}\sum_{n=0}^{\infty}\frac{1}{n\pi\dfrac{b\gamma}{a}+1}\cos\frac{n\pi}{a}xe^{-(n\pi/a)h}. \tag{21}$$

For the density of the protective current we find analogously (using the first of boundary conditions (17)):

$$j|_{y=0} = -\frac{I}{2a}\sum_{n=0}^{\infty}\frac{1}{n\pi\dfrac{b\gamma}{a}+1}\cos\frac{n\pi}{a}xe^{-(n\pi/a)h}. \tag{22}$$

For approximate summation of the series appearing in formula (21), we differentiate it with respect to h and then replace the ratio $\varphi(n) = \dfrac{n}{n\pi\dfrac{h\gamma}{a}+1}$ by the function $\varphi_1(n) = \dfrac{a}{\pi b\gamma}\left(1 - e^{(\pi b\gamma/a)n}\right)$, which agrees closely with $\varphi(n)$ both for large and small values of n. The series thus obtained can be summed, and after integrating with respect to h we find:

$$U|_{y=0} \approx \frac{I}{4\pi\gamma} \ln \frac{\operatorname{ch}\frac{\pi(b\gamma+h)}{a} - \cos\frac{\pi x}{a}}{\operatorname{ch}\frac{\pi h}{a} - \cos\frac{\pi x}{a}}. \tag{23}$$

Analogous computations for the series entering in (22) lead to the expression:

$$j|_{y=0} \simeq -\frac{I}{4\pi b\gamma} \ln \frac{\operatorname{ch}\frac{\pi(b\gamma+h)}{a} - \cos\frac{\pi x}{a}}{\operatorname{ch}\frac{\pi h}{a} - \cos\frac{\pi x}{a}}. \tag{24}$$

Thus, the approximate formulas obtained make it possible to determine the distribution of protective potential and current of the system of protectors under consideration at any point of the section of shielded surface. In particular, putting x = 0, we find the maximal values of these quantities:

$$\left.\begin{aligned} U_{\substack{\max \\ (x=y=0)}} &\simeq \frac{I}{4\pi\gamma} \ln \frac{\operatorname{ch}\frac{\pi(b\gamma+h)}{a} - 1}{\operatorname{ch}\frac{\pi h}{a} - 1} \\ j_{\substack{\max \\ (x=y=0)}} &\simeq \frac{I}{4\pi b\gamma} \ln \frac{\operatorname{ch}\frac{\pi(b\gamma+h)}{a} - 1}{\operatorname{ch}\frac{\pi h}{a} - 1} \end{aligned}\right\} \tag{25}$$

The minimal values of protective potential and current density will be attained at the points x = ±a and the corresponding expressions for them have the form:

$$\left.\begin{aligned} U_{\min} &\simeq \frac{I}{4\pi\gamma} \ln \frac{\operatorname{ch}\frac{\pi(b\gamma+h)}{a} + 1}{\operatorname{ch}\frac{\pi h}{a} + 1} \\ j_{\min} &\simeq \frac{I}{4\pi\gamma b} \ln \frac{\operatorname{ch}\frac{\pi(b\gamma+h)}{a} + 1}{\operatorname{ch}\frac{\pi h}{a} + 1} \end{aligned}\right\} \tag{26}$$

The formulas obtained also make it possible to determine directly the amount of current per unit length of protector necessary for complete protection of a surface section of given area.

For this it is necessary in expressions (26) to take the left members equal to the minimal criterion of electrochemical protection (with respect to potential or current density, respectively) and then solve the equation so obtained with respect to the current.

In conclusion let us consider an application of the results obtained.

Example. Suppose that a flat section of metal (St. 3) surface immersed in sea water is shielded by a protector of spherical shape which is located at a distance of h = 0.8 m from the surface. Let us assume that the current per unit length of protector is equal to I = 2 A and then find the zone of shielding action of this protector, supposing that the current density of cathode polarization on the boundaries of this zone should be not less than

$$j_{\min}^{0} = 0.01 \frac{a}{м^2}.$$

The cathode polarizability of St. 3 on a linear section of the polarization curve is equal to $b \simeq 0.04\ \Omega \cdot m^2$. The conductivity of sea water (of Pacific Ocean composition) we shall assume equal to $\gamma = 4 \frac{1}{\text{ohm}\cdot\text{m}}$. Then using formula (14), we find the radius of shielding action of the protector

$$r_{\text{pro}} = \left[\frac{I(2h + b\gamma)}{4\pi j_{\min}^{\circ}}\right]^{1/3} = \left[\frac{2(1.6 + 4\cdot 0.04)}{4\pi\cdot 0.01}\right]^{1/3} = 5.2\ \text{m}.$$

CONCLUSIONS

1. Expressions have been obtained for the distribution of potential and current on a linearly polarized plane metal surface shielded by a single protector (or anode) which is located at a given distance from this surface which exceeds the dimensions of the protector.

An analogous problem has been solved in two-dimension formulation for the case of an infinite system of protectors which are located at a distance from one another which is commensurate with their distance from the shielded surface.

2. The results obtained are expressed in simple engineering formulas and are illustrated by an example.

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All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. ***Some or all of this periodical literature may well be available in English translation.*** **A complete list of the cover-to-cover English translations appears at the back of this issue.**
