PHYSICS

FORMULATION OF THE EQUATIONS FOR SCATTERING WITH REARRANGEMENT IN THE COORDINATE REPRESENTATION

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In the present article we present in the coordinate representation the derivation of nonsingular integral equations for the problem of scattering with rearrangement with three composite fragments. The relationships that we obtain enable us, in extending the considerations of [1], to clarify the difficulties resulting from attempting to use the adiabatic approximation in the problem with rearrangement, to indicate the necessity for constraints on choosing the effective potentials in the distorted-wave method, and also to provide a simple method for obtaining general threshold equations. Also, by making use of the coordinate representation, we are able to give greater clarity to all the reasoning.

We first consider scattering with rearrangement in a system of three fragments which consist of various particles, at energies for which the only reaction channels that are open are those which correspond to the union of some two fragments in a bound state. We select the zero point for energy E in the system of the total center of mass so that the indicated condition can be expressed as $E \leq 0$. We label the fragments with indices $\alpha = 1, 2, 3$, and we denote their masses by m_α. The reaction channels under consideration also are denoted by the index α, with it being understood that in reaction channel α the fragment α goes off to infinity, and the other two fragments remain connected. In each channel we introduce Jacobi coordinates R_α, $\mathbf{r}_\alpha$ (ζ_α), where R_α is the distance between the center of mass of the fragment α and the common center of gravity of the two remaining fragments; $\mathbf{r}_\alpha$ is the distance between the centers of mass of these two fragments; and the (ζ_α) are the remaining "inner" coordinates, which do not appear below in explicit form and will not be written out. We denote the Hamiltonian of the system by $\mathscr{H} = \mathscr{K} + V = \mathscr{K} + V_\alpha + v_\alpha$, where $\mathscr{K}$ is the kinetic-energy operator, $\mathscr{K} = P_\alpha^2/2M_\alpha + \mathbf{p}_\alpha^2/2m_\alpha$ $(\alpha = 1, 2, 3)$, and v_α denotes that part of the interaction that vanishes (we assume, sufficiently rapidly) as $R_\alpha \to \infty$. Here $\mathbf{P}_\alpha$ and $\mathbf{p}_\alpha$ are momentum operators corresponding to motion along the coordinates R_α and r_α, $M_\alpha = m_\alpha(m_\beta + m_\gamma)/M$, $m_\alpha = m_\beta m_\gamma/(m_\beta + m_\gamma)$, $M = m_1 + m_2 + m_3$ $(\alpha \neq \beta, \gamma; \gamma \neq \alpha, \beta)$. We shall seek the total wave function Ψ of the scattering problem in the form of the sum $\Psi = \Psi_1 + \Psi_2 + \Psi_3$, where each of the functions Ψ_α has a nonzero asymptote of the form required by the scattering problem in "its own" channel only, and has a zero asymptote in the remaining channels, and satisfies the system of equations

$$\begin{aligned}(\mathscr{K} + v_1 - E)\Psi_1 &= -V_1(\Psi_2 + \Psi_3),\\ (\mathscr{K} + v_2 - E)\Psi_2 &= -V_2(\Psi_3 + \Psi_1),\\ (\mathscr{K} + v_3 - E)\Psi_3 &= -V_3(\Psi_1 + \Psi_2).\end{aligned} \tag{1}$$

The separation of the functions Ψ into parts arranged equivalently to the Lippman-Schwinger equation was accomplished by L. D. Faddeev [2], and was performed earlier by Sasakawa [3]. We now show that there is a consistent construction for the functions Ψ_α with the indicated properties which can be done directly in the coordinate representation (rather than in the momentum representation which is usually used). The system (1) has a fundamental advantage in comparison with the Schrödinger equation for Ψ: The sources which appear in (1) are localized in the space of the variables R_α, $\mathbf{r}_\alpha$ over all directions, which allows us to di-

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rectly convert from (1) to "well-behaved" integral equations by using the Green's functions G_α (α = 1, 2, 3), which, in "their own" coordinate representation, have the form

$$\langle \mathbf{R}_\alpha', \mathbf{r}_\alpha' | G_\alpha | \mathbf{R}_\alpha, \mathbf{r}_\alpha \rangle = i \sum_{LMlmn_\alpha} \int_{\mathscr{E}_{0\alpha}}^{\infty} d\mathscr{E}_\alpha \Big\{ \frac{1}{k_\alpha R_\alpha} Y_{LM}\Big(\frac{\mathbf{R}_\alpha}{R_\alpha}\Big) \varphi_\alpha (lmn_\alpha \mathscr{E}_\alpha; \mathbf{r}_\alpha) \times [\theta(R_\alpha - R_\alpha') h^{(1)}_{L+1/2}(k_\alpha R_\alpha) \times j_{L+1/2}(k_\alpha R_\alpha') + \theta(R_\alpha' - R_\alpha) h^{(1)}_{L+1/2}(k_\alpha R_\alpha') j_{L+1/2}(k_\alpha R_\alpha)] \times \frac{1}{R_\alpha'} Y_{LM}^* \Big(\frac{\mathbf{R}_\alpha'}{R_\alpha'}\Big) \varphi_\alpha (lmn_\alpha \mathscr{E}_\alpha; \mathbf{r}_\alpha') \Big\},$$

$$k_\alpha = \sqrt{2M_\alpha(E - \mathscr{E}_\alpha) + i\varepsilon}, \quad \varepsilon \to +0, \tag{2}$$

where $h^{(1)}_{L+1/2}$ and $j_{L+1/2}$ are Riccati-Bessel functions, with $h^{(1)}_{L+1/2}(k_\alpha R_\alpha)$ containing only diverging or damped waves when $R_\alpha \to \infty$; the Y_{LM} are spherical harmonics normalized to unit solid angle; and $\varphi_\alpha(lmn_\alpha \mathscr{E}_\alpha; \mathbf{r}_\alpha)$ are the components of the complete set of solutions of the equation

$$(p_\alpha^2/2m_\alpha + V_\alpha - \mathscr{E}_\alpha)\varphi_\alpha = 0, \tag{3}$$

where $\mathscr{E}_\alpha$ runs through the complete spectrum of values of the two-particle problem,[1] beginning with the lowest value $\mathscr{E}_{0\alpha}$, which exists by assumption; l and m are the total intrinsic angular momentum and the component of the angular momentum; and the n_α are the remaining discrete quantum numbers of the fragments. We make use of the result [4] that the value of the decay threshold energy is the same in all the reaction channels. The system (1) together with the boundary conditions of the scattering problem can be represented in the form of a matrix integral equation with a completely continuous kernel identical with the kernel of the Faddeev equation:

$$\mathbf{\Psi}_i = \Psi_{0i} = GV\Psi_i, \tag{4}$$

where

$$\mathbf{\Psi}_i = \begin{vmatrix} \Psi_{1i} \\ \Psi_{2i} \\ \Psi_{3i} \end{vmatrix}, \quad \mathbf{G}_0 = \begin{vmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \end{vmatrix}, \quad \mathbf{V} = \begin{vmatrix} 0 & V_1 & V_1 \\ V_2 & 0 & V_2 \\ V_3 & V_3 & 0 \end{vmatrix}, \tag{5}$$

the index i corresponds to the specific initial state, which is described by a 3 × 1 column matrix that contains, depending on the statement of the problem, the product of the incident plane or spherical wave and the corresponding bound state in the row corresponding to the initial channel, and is zero in the remaining rows. The solution of Eq. (4) actually satisfies the necessary conditions, since, when $R_\alpha \to \infty$, the second term in the right side contains undamped spherical waves only in the α channel. This follows from the fact that each of the Green's functions comprising the matrix G_0 is multiplied by a localized source, and, therefore, in the limit under consideration (as $R_\alpha \to \infty$), the terms of (2) containing $\theta(R_\alpha' - R_\alpha)$ vanish. In each of the "foreign" channels, the functions Ψ_α asymptotically go to zero, since in this case r_α and R_α go to infinity simultaneously. We must keep in mind that on the physical sheet, positive $E - \mathscr{E}_\alpha$ corresponds the approach to a cut from above, in the Riccati-Bessel functions and for $E - \mathscr{E}_\alpha < 0$ both $h^{(1)}_{L+1/2}$, and $j_{L+1/2}$ behave like $e^{-|k_\alpha|R_\alpha}$. It is important to emphasize that the considerations given about localizing the Ψ_α in their own channels are valid only for $E \leq 0$; for $E > 0$, all the arguments obviously should be substantially modified; it might also become necessary to convert, as in [2, 3], to the momentum representation, since the range of integration $0 < \mathscr{E}_\alpha \leq E$ in the Green's functions (2) after substituting them into (4) will give undamped waves corresponding to the decay channel of the system into three fragments and the "overlapping" of all the remaining channels.

Considering the asymptote of the right side of (4) after substituting from (2), we find that the scattering amplitude in the state $f = \{l_\alpha f, m_\alpha f, n_\alpha f, L_\alpha f, M_\alpha f, \mathscr{E}_\alpha\}$ of the α channel, determined by the coefficient in front of the corresponding asymptotic diverging spherical wave, is represented in the coordinate representation by the expression

$$A_{fi} = \int \Psi_{0f}^* (\mathbf{R}_\alpha^f, \mathbf{r}_\alpha^f) V_{\alpha^f} (\mathbf{r}_\alpha^f) (\Psi_{i\beta} + \Psi_{i\gamma}) d^3\mathbf{R}_\alpha d^3\mathbf{r}_\alpha,$$

$$\Psi_{0f}(\mathbf{R}_\alpha, \mathbf{r}_\alpha) = \frac{1}{k_\alpha R_\alpha} Y_{LM}\Big(\frac{\mathbf{R}_\alpha}{R_\alpha}\Big) j_{L+1/2}(k_\alpha R_\alpha) \varphi_\alpha (l_\alpha, m_\alpha, n_\alpha, \mathscr{E}_\alpha; \mathbf{r}_\alpha) \tag{6}$$

or in matrix notation by

$$A_{fi} = (\mathbf{\Psi}_{0f}, \mathbf{V}\mathbf{\Psi}_i), \tag{7}$$

where the parentheses denote the scalar product, including integration over all coordinates, and Ψ_{0f}

[1]The integral with respect to $d\mathscr{E}_\alpha$ in (2) includes summation over the discrete spectrum and integration over the continuous spectrum.

is a column matrix with one nonzero row corresponding to the final channel.

We introduce a symbolic solution of (4), writing it in the form of a column matrix Ψ_i:

$$\Psi_i = \Psi_{0i} - GV\Psi_{0i}, \tag{8}$$

where G is the solution of the equation with a "well-behaved" kernel (because the matrix V has no diagonal elements):

$$G = G_0 - G_0VG. \tag{9}$$

By using (8), we can represent A_{fi} as

$$A_{fi} = (\Psi_{0f}, (V - VGV)\Psi_{0i}). \tag{10}$$

Note that in our derivation of (10), we never used the orthogonality of the wave functions of different channels. It is also interesting to note that (10) contains terms which appear in the Gell-Mann–Goldberger equation [5] in rearranged form. If we wish to speak in the language of Weinberg graphs [6], then the last (furthest left) interaction in the notation of the expressions for the amplitude is the interaction responsible for the formation of the bound state.

All the preceding considerations can be extended simply to the case of a larger number of fragments, and also to the case of identical particles in different fragments. In the latter case, the number of reaction channels must be increased to correspond to the possible number of rearrangements of identical particles between fragments, with an increase in the number of equations in the system (1). The subsequent symmetrization enables us to reduce the number of equations; nonlocal interactions containing a rearrangement operator will enter into the new system. It is clear that a construction similar to that carried out above is possible only for a definite separation of the interactions in the different channels. Thus, the present approach, like the general rearrangement of L. D. Faddeev, does not allow, as was noted in [1], an expansion of the functions Ψ in so-called adiabatic functions, i.e., eigenfunctions of the complete Hamiltonian, from which we exclude some part of the kinetic-energy operator. The latter expansion is used especially frequently in atomic-molecular problems without requiring reduction to a system of equations with a "well-behaved" kernel.

The separation of the interactions, on which the notation of Eqs. (1) is based, is not unique. For example, we can add to V_1 any effective decreasing-with-distance interaction of the form $W(R_1)P_1$, where P_1 is a projection operator on the eigenfunctions (3) corresponding to the discrete spectrum in channel 1; at the same time, we must substract this interaction from V_2 and (or) V_3. The interaction $W(R_1)$ should not cause rearrangements and accordingly depends only on R_1. Our entire treatment will hold unchanged if we merely make a change in the functions from which the Green's function is constructed in the coordinate representation. We must introduce the operator P_1 in order to conserve the local character of the sources in (1). Analogously, we can obtain equations similar to the equations using the method of distorted waves, and we can carry out a generalized threshold consideration, taking into account the long-range part of the potential. This will be done in a future paper.

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