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FLUID MECHANICS

MOTION OF PARTICLES SUSPENDED IN TURBULENT FLOW

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We consider the motion of solid particles with respect to the turbulent gas in which they are suspended. The results obtained can also be applied to the motion of solid particles, undeformed droplets, and bubbles in a liquid, and to the motion of small liquid droplets in a turbulent gas. We consider only the case in which the specific volume γ occupied by the particles is small compared to unity.

We neglect the influence of the particles on the motion of the gas. The range of applicability of our results will be investigated in detail in a subsequent paper. The turbulent flow is assumed to be locally homogeneous, isotropic, and stationary, and the dimension R of the suspended particles is assumed to be so small that the Reynolds number for the relative motion of the particles in the gas is small compared with unity. The equation of motion of particles satisfying this condition was obtained by Chen and then made more precise by Corsin and Lamley [1]. This equation relates the velocity W_i of a particle in a gas to the velocity of motion U_i of the gas. In general form it is a nonlinear partial differential equation. It can however be transformed into a linear integro-differential equation with the time as the only independent variable if

$$\frac{R^2}{\nu}\frac{\partial U}{\partial x} \ll 1, \quad \frac{W}{\nu}\frac{\partial U}{\partial x} \Big/ \frac{\partial^2 U}{\partial x^2} \gg 1,$$

where ν is the kinematic viscosity of the gas.

Let U_λ and ω_λ be the velocity and frequency of pulsations with dimensions λ. Using the fact that the particle velocity is close to the velocity U_L of the largest oscillations of the gas, which are of a scale L of order of the dimensions of the system, we have

$$\frac{W}{\nu}\frac{\partial U}{\partial x} \Big/ \frac{\partial^2 U}{\partial x^2} \sim \frac{W}{\nu}\frac{U_\lambda}{\lambda} \Big/ \frac{U_\lambda}{\lambda^2} \sim \frac{U_L \lambda}{\nu} > \frac{U_L L}{\nu}\frac{l}{L} \sim \mathrm{Re}_L^{1/4} \gg 1,$$
$$\frac{R^2}{\nu}\frac{\partial U}{\partial x} \sim \frac{R^2}{\nu}\frac{U_\lambda}{\lambda} \sim \frac{R^2}{\nu}\omega_\lambda < \frac{R^2}{\nu}\frac{\nu}{l^2} \sim \frac{R^2}{l^2} < 1.$$

Hence, if $R < l$ the equation of steady relative particle motion is

$$\frac{dV_i}{dt} = (\alpha - 1)\frac{dU_i}{dt} - \beta V_i - \sqrt{\frac{3\alpha\beta}{\pi}} \int_{-\infty}^{t} \frac{dV_i}{d\tau}\frac{d\tau}{\sqrt{t-\tau}} + g_i. \quad (1)$$

The notation in (1) is as follows: $V_i = W_i - U_i$ is the velocity of a particle with respect to the gas, $\alpha = 3\rho_0/(2\rho + \rho_0)$, where ρ_0 is the gas density and ρ is the particle density, g_i is the acceleration of free fall, and $\beta = k_1\alpha\nu/R^2$ is the characteristic frequency determining the motion of the particles in the gas and is equal to the reciprocal of the time required for the gas flowing past a particle to alter its flow state. The quantity k_1, and also the quantities $k_2, k_3, \ldots$, which appear later, are numerical constants of order unity. Since (1) is linear, the velocity V_i is the superposition of two independent quantities: the velocity due to gravity and the velocity related to the motion of the gas. Since here we are interested in the particle motion caused by the movement of the gas, the term g_i can be omitted from (1). However, since this equation is linear, our results remain valid in the presence of gravity. Under our assumption $R < l$, (1) can be further simplified. It can be shown [2] that the term involving the integral is small compared to the remaining terms if $\omega_\lambda R^2/\nu \ll 1$ for all frequencies of the turbulence spectrum. It is easily shown that this inequality is equivalent to the inequality $R^2 < l^2$. In fact

$$\omega_\lambda R^2/\nu < \omega_l R^2/\nu \sim R^2/l^2.$$

Hence, under our assumptions the equation for the relative motion of a particle in the gas is

$$\frac{dV_i}{dt} = (\alpha - 1)\frac{dU_i}{dt} - \beta V_i. \quad (2)$$

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In turbulent flow the velocities U_i and V_i are random functions of the time. To determine the properties of the motion of particles with respect to the gas we must know the relation between the Lagrange time correlation of U_i and the Lagrange time correlation of V_i. We therefore determine the relation between the correlations

$$Q = \overline{U_i(t)\,U_i(t+\tau)}, \quad S = \overline{V_i(t)\,V_i(t+\tau)}.$$

A bar over a quantity indicates that it is to be averaged over all states of the system. Since we assume that the motion is stationary, Q and S are functions only of the correlation time τ and are independent of t.

To find the relation between $Q(\tau)$ and $S(\tau)$, we express U_i and V_i as Fourier integrals and then, using (2), we determine the relation between the spectrum $\Phi(\omega)$ of the velocity field V_i and the spectrum $\Psi(\omega)$ of the velocity field U_i:

$$\Phi(\omega) = (1-\alpha)^2 \frac{\omega^2}{\omega^2+\beta^2} \Psi(\omega). \qquad (3)$$

Since the correlation is simply the Fourier transform of the corresponding spectrum, (3) implies that

$$S(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\omega)\, e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(1-\alpha)^2\omega^2}{\omega^2+\beta^2} \Psi(\omega)\, e^{i\omega\tau} d\omega. \qquad (4)$$

To establish the relation between the integral in (4) and the correlation function $Q(\tau)$, we must use Parseval's formula. We obtain

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\omega^2}{\omega^2+\beta^2} e^{i\omega\tau} d\omega = \delta(\tau) - \frac{\beta}{2} e^{-\beta(\tau)}$$

and, since

$$\int_{-\infty}^{\infty} \Psi(\omega)\, e^{i\omega\tau} d\omega = Q(\tau),$$

we have

$$S(\tau) = (1-\alpha)^2 \left[Q(\tau) - \frac{\beta}{2} \int_{-\infty}^{\infty} Q(\zeta)\, e^{-\beta|\tau-\zeta|} d\zeta \right]. \qquad (5)$$

The evenness of $Q(\zeta)$ implies that (5) can be written

$$S(\tau) = (1-\alpha)^2 \Big[Q(\tau)$$

$$- \frac{\beta}{2} \int_{0}^{\infty} Q(\zeta)\left(e^{-\beta(\tau+\zeta)} + e^{-\beta(\tau-\zeta)}\right) d\zeta \Big]. \qquad (6)$$

It is plain from (6) that we must know the time correlation of the gas velocity in order to be able to find $S(\tau)$. From our assumption that the presence of particles has only a negligible influence on the gas motion, it follows that we may use in (6) the correlation function $Q(\tau)$ of gas not containing particles. There is no general theoretical formula for $Q(\tau)$ applicable for all correlation times. We can, however, use an empirical formula which approximates the behavior of the correlation functions. For large Reynolds numbers we can use the exponential formula given in [3, 4]. Hence, if we assume that $Q(\tau) = Q(0)\exp\{-\omega_L\tau\}$, (6) yields

$$S(\tau) = (1-\alpha)^2 Q(0) \frac{\omega_L^2}{\omega_L^2-\beta^2} \left[e^{-\omega_L\tau} - \frac{\beta}{\omega_L} e^{-\beta\tau} \right]. \qquad (7)$$

Knowing the correlation function of relative particle motion, we can determine the characteristics of this motion with respect to the gas. The most important parameters are the mean-square velocity $(\overline{V^2})^{1/2}$ and the mean oscillation amplitude a. It follows from the correlation function that $Q(0) = \overline{U^2}$ and $S(0) = \overline{V^2}$; hence,

$$\overline{V^2} = \overline{U^2}\frac{(1-\alpha)^2\omega_L}{\omega_L+\beta} \approx U_L^2 \frac{(1-\alpha)^2\omega_L}{\omega_L+\beta}. \qquad (8)$$

It follows immediately from (8) that if the particles are suspended in the gas flow ($\alpha \sim 10^{-3}$) and they are not too small (so that $\beta \ll \omega_L$), the mean-square velocity of the relative particle motion coincides with the mean turbulence pulsation velocity of the gas. If the particles are suspended in a turbulent liquid, we always have $\beta \gg \omega_L$ and the mean-square velocity of the relative motion $(\overline{V^2})^{1/2}$ is always much smaller than the mean-square velocity of the liquid $(\overline{U^2})^{1/2}$.

All frequencies of the turbulence spectrum influence particle motion in turbulent flow, and so it is clear that the mean amplitude of particle oscillation must be taken to be the mean-square displacement of a particle with respect to the gas $[\overline{r^2(t)}]^{1/2}$ after a sufficiently long elapsed time, i.e., for $t \to \infty$. In view of the fact that r_i and V_i are related by the equation $r_i(t) = \int_0^t V_i(t')\,dt'$ and the random function $V_i(t)$ is stationary, we can express the mean-square displacement $[\overline{r^2(t)}]^{1/2}$ in terms of the correlation function $S(\tau)$ [5]:

$$\overline{r^2(t)} = 2\int_0^t (t-\tau)\, S(\tau)\, d\tau. \qquad (9)$$

Even though the function $V_i(t)$ is stationary, the mean-square particle displacement is clearly a function of the time. This fact is related to the fact that the integral of a stationary random function is not in general a stationary function [5]. Substituting the expressions for $S(\tau)$ obtained from (7) in (9) and integrating, we have

$$\overline{r^2(t)} = (1-\alpha)^2 \frac{\overline{2U^2}}{\omega_L^2-\beta^2}$$

$$\times\left[\frac{\omega_L}{\beta}\left(1-e^{-\;t}\right)-\left(1-e^{-\omega_L t}\right)\right], \tag{10}$$

$$a^2 = \lim_{t\to\infty} r^2(t) = \frac{\overline{2U^2}}{\beta(\omega_L+\beta)} \approx \frac{2U_L^2}{\beta(\omega_L+\beta)}. \tag{11}$$

A comparison of the quantity a with the dimension of pulsations with observed scale $L \sim U_L/\omega_L$ yields

$$a/L \sim \omega_L/\sqrt{\beta(\omega_L+\beta)}. \tag{12}$$

Equation (12) shows that, in the case of particle motion in a gas flow with the condition $\overline{V^2} = \overline{U^2}$, the quantity a can considerably exceed L. In the opposite limiting case $\beta \gg \omega_L$, the mean-square particle displacement a is much smaller than the dimension L of the largest turbulent pulsations. In this case, after a sufficiently long time, the motion of a particle suspended in a turbulent flow can be considered to be the superposition of motion due to its entrainment by large vortices and random motion with respect to the gas or liquid inside a region with dimensions of order a.

LITERATURE CITED

1. S. Corsin and J. Lamley, Appl. Sci. Res., 6A, 114 (1956).
2. L. D. Landau and E. M. Lifshits, Mechanics of Continuous Media [in Russian] (1953).
3. G. Taylor, Proc. London Math. Soc., 20, 196 (1921).
4. A. A. Kalinske and C. L. Pien, Ind. and Eng. Chem., 36, 220 (1944).
5. A. A. Sveshnikov, Applied Random Function Theory [in Russian] (1961).