CALCULATION OF PARAMETERS IN THE WAVE FLOW REGIME OF THIN LIQUID FILMS

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An appropriate method for solving the equations of film flow hydrodynamics was developed in [1]. This method is theoretically applicable to regimes with any length and amplitude of waves. Since complete solution using this method involves a tedious numerical solution using a digital computer, the limiting case of waves of small amplitude was investigated in [1]. This case can be solved analytically by the method of perturbation of the parameter $A = (h_{max} - h_0)/h_0$, where h_{max} and h_0 are the maximum and average film thikcness (along the length of the wave), respectively. The wavelength and phase velocity can be calculated as a function of Reynolds and Weber numbers, but the amplitude A, which serves as the small parameter, cannot be determined. The direct variation method of Galerkin can be used to solve the problem of the wave regime with a finite amplitude.

Let us examine a liquid film, flowing down a flat vertical wall y = 0 under the force of gravity. We will use a rectangular system of coordinates with axis y perpendicular to the wall and axis x directed in the direction of film flow, i.e., vertically down. We will see a solution to the hydrodynamic problem which corresponds to a regular unattenuated wave regime of flow. In this case independent variables x and t (time) will appear in the solution in the form x-ct, where c is the phase velocity of the waves. We will approximate the x component of velocity $u(\xi, y)$ by the expression

$$u(\xi, y) = v_0[a_0(\xi) \quad (y/h_0) + a_1(\xi) \quad (y^2/h_0^2)], \tag{1}$$

where $\xi = (x - \alpha v_0 t)/h_0$ is the dimensionless axial coordinate in the system moving with a wave phase velocity $c = \alpha v_0$ (v_0 is the average downflow velocity of the film), and h_0 is the average film thickness. The corresponding expression for the y component of velocity $v(\xi, y)$ can be written using the continuity equation. Integrating the Navier-Stokes equation along coordinate y, after the expression for u and v, and using the condition of continuity of tangential forces at the film surface $y = h(\xi)h_0$ and the condition of conservation of integrated liquid flow in the film, leads to the following system of equations for determining the functions $a_0(\xi)$, $a_1(\xi)$, and $h(\xi)$:

$$h\left\{\frac{1}{\mathrm{We}\,(1+h'^2)^{3/2}} - \frac{2}{\mathrm{Re}}\left[a_0 h + a_1 h^2 + \frac{\alpha}{2(1+h'^2)}\right]\right\}'' + \sum_{k=0}^{4} \frac{L_k h^{k+1}}{k+1} + \sum_{k=0}^{5} \frac{(M_k h^{k+2})'}{k+2} = 0; \tag{2}$$

$$(1-h'^2)(a_0+2a_1h) - 2h'(a_0'h + a_1'h^2) - h'^2h''(a_0h + a_1h^2)/(1+h'^2) + \alpha h''(h'^4 + 2h'^2 - 1)/(1+h'^2)^2 = 0; \tag{3}$$

$$a_0h^2/2 + a_1h^3/3 - \alpha h + \alpha + 1 = 0. \tag{4}$$

The primed terms represent derivatives with respect to ξ, while the symbols L_k and M_k represent the following combinations of coefficients a_0 and a_1 and their derivatives:

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$$L_0 = -gh_0^3 / \nu^2 \mathrm{Re}^2 - 2a_1 / \mathrm{Re}; \quad L_1 = -\alpha a_0' - a_0'' / \mathrm{Re};$$
$$L_2 = -\alpha a_1' + a_0 a_0' / 2 - a_1'' / \mathrm{Re};$$
$$L_3 = {}^2/_3 a_0 a_1'; \quad L_4 = {}^1/_3 a_1 a_1';$$
$$M_0 = a_0' / \mathrm{Re}; \quad M_1 = 2a_1' / \mathrm{Re}; \quad M_2 = \alpha a_0'' / 2 + a_0''' / 2\mathrm{Re};$$
$$M_3 = \alpha a_1'' / 3 - a_0 a_0'' / 2 + a_0'^2 / 2 + a_1''' / 3\mathrm{Re};$$
$$M_4 = -a_0 a_1'' / 3 - a_0'' a_1 / 2 + 5a_0' a_1' / 6;$$
$$M_5 = -a_1 a_1'' / 3 + a_1'^2 / 3.$$

The Reynolds number Re and Weber number We, which appear in these relations, are calculated as follows: $\mathrm{Re} = v_0 h_0 / \nu$ and $\mathrm{We} = \rho\, \nu^2 \mathrm{Re}^2 / \sigma\, h_0$, where ν and ρ are the kinematic viscosity and density of the liquid and σ is the surface tension at the liquid-gas interphase.

Let us expand functions a_0, a_1, and h into Fourier series:

$$a_0 = a_{00} + a_{01} \sin n\xi + \bar{a}_{01} \cos n\xi + a_{02} \sin 2n\xi + \bar{a}_{02} \cos 2n\xi + \ldots; \tag{5}$$

$$a_1 = a_{10} + a_{11} \sin n\xi + \bar{a}_{11} \cos n\xi + a_{12} \sin 2n\xi + \bar{a}_{12} \cos 2n\xi + \ldots; \tag{6}$$

$$h = 1 + A \sin n\xi + h_2 \sin 2n\xi + \bar{h}_2 \cos 2n\xi + \ldots. \tag{7}$$

We will approximate series (5)-(7) by expressions containing a finite number of harmonics in order to obtain an approximate solution to the problem. In this case the number of equations, obtained by substituting (5)-(7) into (2)-(4) and equating the coefficient of each harmonic to zero, will be one less than the number of unknowns, and it is necessary to find an additional physical condition for complete solution. Reynolds' [2] stability principle, according to which the rate of change of integral kinetic energy of the liquid under any disturbance of the main flow must not be positive, can be used as such a condition.

The integral form of the energy transfer equation is [1]

$$\frac{dE_{\text{кин}}}{dt} \equiv \frac{\rho}{2} \int_0^\lambda \int_0^h \frac{\partial \nu^2}{\partial t}\, dy\, dx = \int_0^\lambda \int_0^h \Big\{ \rho g u - \frac{\partial}{\partial x}\Big[u\Big(\frac{\rho \nu^2}{2} + p\Big)\Big] - \frac{\partial}{\partial y}\Big[v\Big(\frac{\rho \nu^2}{2} + p\Big)\Big] + \rho\nu (u\, \Delta u + v\, \Delta v) \Big\}\, dy\, dx. \tag{8}$$

Let us take the steady-state wave movement, described by Eqs. (1)-(4), as the main stream. We will examine the velocity field v* which occurs as a result of superposing some infinitesimal disturbance on the main stream, and will assume that this disturbance is periodic and has the same phase velocity and wavelength as the main stream. Let A* represent the amplitude of the resultant disturbed movement. Obviously when the disturbance is infinitesimally small A* can be represented as the sum $A^* = A + \chi$, where $\chi \ll A$. If Eq. (8) is written in functional form

$$dE_{\text{kin}} / dt = \Phi[\mathrm{v}^*(A^*)], \tag{9}$$

the right-hand side of (9) can be expressed as an expansion, based on the condition that $\chi \ll A$:

$$\Phi[\mathrm{v}^*(A^*)] = \Phi[\mathrm{v}(A)] + \chi \frac{d\Phi}{dA}[\mathrm{v}(A)] + \ldots, \tag{10}$$

where v(A) is the steady-state field of velocities described by Eqs. (1)-(4). The second term of the right-hand side of (10) represents the contribution of the infinitesimal disturbance to the energy balance. This contribution can be either positive or negative in the general case, i.e., at an arbitrary amplitude A and an arbitrary sign of amplification χ (positive or negative). Since, by Reynolds' stability criteria, a disturbance which contributes positively to function (9) results in a less stable flow regime, it is obvious that the most stable regime will be that for which function $\Phi[\mathrm{v}(A)]$ will have a minimum with respect to variation in amplitude A:

$$\frac{d\Phi}{dA}[\mathrm{v}(A)] = 0. \tag{11}$$

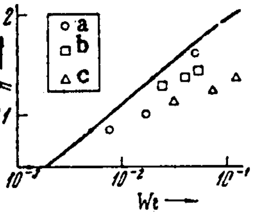

Fig. 1

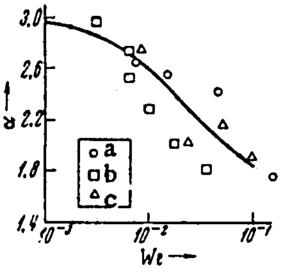

Fig. 2

Fig. 1. Wave number n as a function of Weber number for water films. The solid line represents the calculated curve. Experimental points: a) data from [6]; b) data from [7]; and c) data from [4].

Fig. 2. Phase velocity α as a function of Weber number for water films. Designations the same as in Fig. 1.

In this case the contribution of an infinitesimally small disturbance to function (9) will be two orders of magnitude smaller than χ and, consequently, will have no effect on the stability of flow. Thus condition (11) determines that value of amplitude A at which a stable wave regime of flow is established according to Reynolds criteria. It should be noted that at Reynolds numbers approaching zero, when the main contribution to the energy balance (8) is by viscous forces, condition (11) is equivalent to the condition of minimum energy dissipation [3, 4].

The coefficients a_{0k}, $\bar{a}_{0k}$, a_{1k}, $\bar{a}_{1k}$, h_k, and $\bar{h}_k$, which appear in the Fourier series (5)-(7), and also the average film thickness h_0 and wave parameters n, α, and A can be determined as follows. Using Galerkin's method and approximating series (5)-(7) by a finite number $(2j+1)$ of terms containing j harmonics, we substitute the approximate expressions for a_0, a_1, and h into Eqs. (2)-(4) and regroup the terms of these equations so that the left-hand sides are also in the form of Fourier series. Equating the coefficients of each harmonic in the transformed Eqs. (2)-(4) to zero, we can obtain a system of equations to determine the desired coefficients.* However, due to the complexity of Eqs. (2)-(4), it is not practical to perform the analytic sequence of operations (which, by Galerkin's method, leads to a system for determining a finite number of coefficients) even when series (5)-(7) is approximated by only two harmonics. Therefore, this is an ideal application for a fast computing machine, and the indicated sequence should be replaced by the following procedure which is equivalent to it. We substitute the approximate expressions, containing j harmonics, for series (5)-(7) into Eqs. (2)-(4). We will represent the resulting equation by regrouping terms into the form

$$F_i = F_{i0} + F_{i1}\sin n\xi + \bar{F}_{i1}\cos n\xi + F_{i2}\sin 2n\xi + \bar{F}_{i2}\cos 2n\xi + \dots \quad (i = 1, 2, 3). \tag{12}$$

Coefficients F_{ik} are functions of the desired values a_{0k}, $\bar{a}_{0k}$, etc. According to Galerkin's method, the true values of these parameters must result in values of zero for coefficients F_{ik} for the first j harmonics of each of the series in (12). We will first determine coefficients F_{ik} assuming arbitrary values for parameters a_{0k}, $\bar{a}_{0k}$, etc., and selecting some fixed array of their values. For this we assume $2m_i+1$ values of argument ξ, where m_i is the number of harmonics not equal to zero in series (12), and calculate the right-hand side of Eq. (12) for each ξ. We thus obtain $2m_i+1$ linear equations in coefficients F_{ik} for each m_i, i.e., a total of $2m_1+2m_2+2m_3+3$ equations. We can find coefficients for the first j harmonics appearing in Eq. (12) from these equations. Using Newton's iterative method of tangents we can find, for any fixed values of Reynolds numbers and amplitudes, a combination of parameters a_{0k}, $\bar{a}_{0k}$, a_{1k}, a_{1k}, h_k, $\bar{h}_k$, n, α, and h_0 which will result in zero values for the coefficients F_{ik}. As was demonstrated above, amplitude A can be found from the condition of the minimum of function (8). If the functions a_0, a_1, and h, found by this iterative procedure, are substituted into the right-hand side of (8) and it is integrated with respect to ξ, we get an expression for dE_{kin}/dt as a function of A. We obtain the desired value of the amplitude by numerical determination of the minimum of this function.

*This procedure was followed in [5], which investigated the flow of long waves.

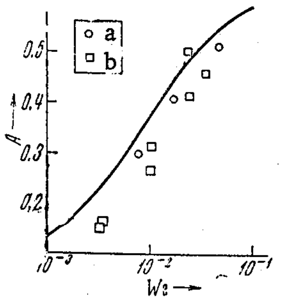

Fig. 3. Amplitude A as a function of Weber number for water films. Designations the same as in Fig. 1.

The above sequence of operations was programmed for solution on a Ts-VM M-220 computer. The calculation is carried out using the first two harmonics, i.e., j = 2. The results of these calculations are shown in Figs. 1-3. Experimental data, obtained very recently [6], as well as the results of earlier investigations [4, 7], are also shown on these figures. It is evident from a comparison of these curves that the numerical method derived above adequately describes actual wave regimes.

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