WAVE CONDITIONS IN THE FLOW OF THIN FILMS OF LIQUID

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UDC 66.048

Wave-type conditions in the flow of thin films of liquid over the surface of a solid body have been for many years the object of intensive experimental and theoretical study, mainly in connection with the fact that this set of conditions is realized in many types of industrial apparatus (film-type rectifiers and absorbers, columns with plane-parallel packing, etc.). The first attempt to create a quantitative theory of the wave motion of films was made by P. L. Kapitsa [1]. The starting premise of the Kapitsa theory was the assumption of the smallness of the ratio of the mean thickness of the film to the wave length. On the basis of this assumption, to describe the velocity field in the film, use was made of the system of equations of the boundary layer, and a solution of this system was found in the form of a Nusselt profile, with a mean velocity depending on the coordinate in the direction of the motion. Among the theory of film flow which have appeared following [1], a special place is occupied by works devoted to investigation of the stability of different types of laminar flow of thin films with respect to small perturbations of the surface [2-8]. The main conclusion of the above investigations is that all perturbations with sufficiently large (compared to the mean thickness of the film) wave lengths grow with motion downstream; their rate of growth increases with an increase in the Reynolds number. Within the framework of the nonlinear theory of perturbations [8] there has been established the existence of particular wave conditions which are stable with respect to perturbations close in their parameters to the initial wave-type flow. Not long ago an attempt was made to calculate the parameters of the wave flow of films in the case of arbitrary (not necessarily large compared to the thickness) wave lengths [9]. Within the framework of a linearized system of Navier-Stokes equations, there was demonstrated in [9] the possibility of the existence of circulating flows of liquid within the film. The idea of the presence of such flows (with opposing direction of circulation under the crests and in the troughs of the waves) had already been expressed in [1], and then repeatedly used in the literature for the interpretation of experimental data. Circulating flows, if such exist, should play an important role in mass and heat transfer through the film, thanks to the additional mixing which they set up. For this reason, a theoretical description of wave-type conditions with wave lengths and amplitudes comparable to the thickness of the film (only under these conditions is the existence of circulations possible) is of great practical interest. Unfortunately, the consideration of the question in [9], is based on a contradictory system of equations and boundary conditions. This contradictory nature of the system becomes evident with a detailed consideration of an exact system of equations and boundary conditions for film-type flow. Actually, let us consider the boundary conditions on the surface of a film of thickness h(x, t), flowing under the action of the force of gravity along a vertical wall y = 0 (th y axis is directed toward the side of the liquid, and the x axis vertically downward) and coming into content with a motionless ideal gas, the pressure in which will be assumed equal to zero. In the case of a constant surface tension, σ, these conditions have the form [10]

$$p(1+b^2)^2+\sigma\frac{\partial b}{\partial x}(1+b^2)^{1/2}$$
$$+2\mu\left[b(1+b^2)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}-u\frac{\partial b}{\partial x}\right)-(1-b^4)\frac{\partial v}{\partial y}-b^2\frac{\partial b}{\partial x}(v-bu)\right]=0; \tag{1}$$

$$(1-b^4)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+4b(1+b^2)\frac{\partial v}{\partial y}-\frac{\partial b}{\partial x}(u+2bv-ub^2)=0, \tag{2}$$

where μ is the dynamic viscosity of the liquid; $b = \partial h/\partial x$; and u and v designate the components of the velocity with respect to x and y, respectively.

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Institute of Electrochemistry, Academy of Sciences of the USSR. Translated from Doklady Akademii Nauk SSSR, Vol. 183, No. 6, pp. 1388-1391, December, 1968. Original article submitted August 16, 1968.

It follows from (1) and (2) that, in the linear approximation with respect to the dimensionless amplitude of the waves,

$$A = (h_{\max} - h_0) / h_0, \tag{3}$$

where h_{max} and h_0 are, respectively, the maximum and the mean (along the length of a wave) thickness of the film, the boundary conditions for the normal and tangential stresses at the surface of the film have the form

$$p + \sigma \partial^2 h / \partial x^2 + 2\mu (b\, \partial u / \partial y - \partial v / \partial y) = 0; \tag{4}$$

$$\partial u / \partial y + \partial v / \partial x - u\, \partial^2 h / \partial x^2 = 0. \tag{5}$$

In [9] the corresponding conditions are written in the form $p + \sigma\partial^2 h/\partial x^2 = 0$, $\partial u/\partial y = 0$. However, since, in accordance with the equation of continuity, $v \sim (hu/\lambda) \sim u$, all the terms in (4) and (5) are of an identical order of magnitude. In addition, in the case of arbitrary (not necessarily large) wave lengths, it is necessary not only to use the equation for u in its full form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g, \tag{6}$$

but also to take account of all the terms in the equation for v

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{7}$$

In actuality, since, in accordance with (6), $(\partial p/\partial x) \sim \mu\partial^2 u/\partial^2 x$ (or $p \sim \mu u/\lambda$), and $v \sim (hu/\lambda) \sim u$, then the term $\partial p/\rho\partial y$ in Eq. (7) will have the same order or magnitude as all the other terms and there is no basis for separately assuming it equal to zero, as was done in [9].

A correct solution of the problem of the wave motion of thin films can be obtained using a method analogous to the method of moments in the theory of a laminar boundary layer. Let us approximate the local profile u(x, y, t) by the series of functions

$$u_N(x, y, t) = \sum_{k=0}^{N} a_k(\xi)\,(y/h_0)^{k+1} \quad (N = 1, 2, \ldots), \tag{8}$$

where $\xi \equiv (x-ct)/h_0$; c is the phase velocity; $a_k(\xi)$ are unkown functions which, for each fixed value of N, can be determined in the following manner. From the equation of continuity (4c) we find the function $v_N(x, y, t)$

$$v_N(x, y, t) = -\sum_{k=0}^{N} \frac{da_k(\xi)}{d\xi}\,\frac{(y/h_0)^{k+2}}{k+2}. \tag{9}$$

Here, as can easily be seen, functions (8) and (9) automatically satisfy the condition of the adherence of the liquid to the solid wall (y = 0). We substitute (8) and (9) into Eq. (7), and integrate the latter with respect to y within the limits from some arbitrary value of y to y = h. Then, using boundary condition (1), we find the distribution of p(x, y, t) and substitute this distribution, as well as expressions (8) and (9) into Eq. (6). As a result, we obtain an equation containing N + 2 unknown functions $a_0(\xi)$, $a_1(\xi)$, . . ., $a_N(\xi)$ and $h(\xi)$. We obtain a closed system of N + 2 equations for finding these functions by multiplying Eq. (5) successively by 1, y, y^2, . . ., y^{N-1} and integrating the result of the multiplication with respect to y from y = 0 to y = h (in this case, N equations are obtained), and using boundary condition (2) and the condition of the macroscopic material balance in the film:

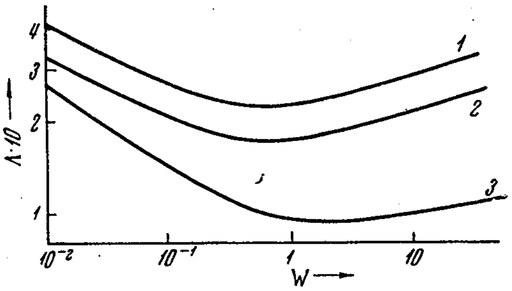

Fig. 1. Dimensionless wave length $\Lambda = (\lambda/2\pi)(\rho g/\sigma)^{1/2}$ as a function of the Weber number for fixed values of the surface tension, the density, and the viscosity: 1) σ = 24 dyn/cm, ρ = 0.8 g/cm^3, $\mu = 1.1 \cdot 10^{-2}$ g/(cm·sec); 2) σ = 72 dyn/cm, ρ = 1 g/cm , $\mu = 10^{-2}$ g/(cm·sec); 3) curve obtained in [9].

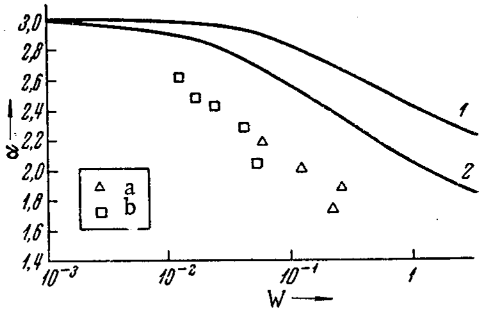

Fig. 2. Dimensionless phase velocity as a function of the Weber number: 1) theoretical curve obtained in [9]; 2) curve obtained in present work. Experimental points: a) water [9]; b) alcohol [11].

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^h u\,dy = 0. \tag{10}$$

In the present work we limit ourselves to finding an approximate analytical solution to the problem of the periodic wave flow of films in approximation by a parabolic form of the local velocity profile (N = 1) and in a linear approximation with respect to the amplitude A. We shall seek the solution of the problem determined by the system of Eqs. (6)-(7) and boundary conditions (4) and (5), in the form

$$u(\xi, y) = a_0(\xi)(y/h_0) + a_1(\xi)(y/h_0)^2; \tag{11}$$

$$v(\xi, y) = -a_0'(\xi)(y/h_0\sqrt{2})^2 - a_1'(\xi)(y/h_0\sqrt{3})^3; \tag{12}$$

$$h(\xi) = h_0[1 + A\varphi(\xi)], \tag{13}$$

where the prime indicated differentiation with respect to ξ. Obtaining, by the method described above, a system of equations for the functions $a_0(\xi)$, $a_1(\xi)$ and $\varphi(\xi)$, and by eliminating the functions a_0 and a_1 from this system, we find, in a linear approximation with respect to A,

$$\varphi(\xi) = \sin n\xi, \tag{14}$$

where $n = 2\pi h_0/\lambda$ is the dimensionless wave number, connected with the mean flow velocity of the film, v_0, with the mean thickness of the film, h_0, with the Reynolds number $Re = v_0 h_0 \rho/\mu$, with the Weber number $W = \rho h_0 v_0^2/\sigma$, and with the dimensionless phase velocity $\alpha = c/v_0$ by a system of algebraic equations:

$$h_0^3 = 3\mu^2 Re/\rho^2 g; \tag{15}$$

$$3\alpha - 9 + n^2(5\alpha - {}^9/_2) + n^4(\alpha/40 - {}^9/_{20}) + \alpha n^6/80 = 0; \tag{16}$$

$$\alpha^2 - 12\alpha/5 + {}^6/_5 + n^2(11\alpha^2/40 - 219\alpha/280 + {}^{87}/_{140} - 1/W) + n^4(\alpha^2/80 - 29\alpha/560) = 0. \tag{17}$$

Figures 1 and 2 give curves, calculated using Eqs. (15)-(17), for the dependence of the dimensionless wave length $\Lambda = (\lambda/2\pi)(\rho g/\sigma)^{1/2}$ and the dimensionless phase velocity, α, on the Weber number for two liquids: water (σ = 72 dyn/cm, ρ = 1 g/cm^3, $\mu = 10^{-2}$ g/(cm·sec); and ethyl alcohol [σ = 23 dyn/cm, ρ = 0.8 g/cm^3, $\mu = 1.1 \cdot 10^{-2}$ g/(cm·sec)]. There follows from these curves, in particular, the existence of a minimum on the curve of $\Lambda(W)$, as well as the presence of a monotonic decrease of α with a rise in Re (that is, with an increase in the irrigation density) for an arbitrarily fixed value of σ. Both the above laws are observed in experiment [11, 12].

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