NUMERICAL METHOD OF CALCULATING THE TURBULENT BOUNDARY LAYER IN A GAS FLOWING AROUND A LAMINAR FILM OF LIQUID

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The main obstacle to the development of accurate methods of calculations of process in industrial chemical equipment is the lack of a qualitative theory of turbulence, in particular, of a theory of turbulent motion near moving interphase boundaries. Existing theories start from one hypothesis or another as to the connection between the turbulent friction and the velocity gradients. However, the systems of equations describing turbulent transfer, even in the case of a given friction law, are so cumbersome and complicated that to obtain visible results recourse must be had to supplementary simplifications which often do not correspond to the real conditions. As a result, comparison of theory with experiment does not always giva a satisfactory answer to the question of the adequacy of the hypothesis assumed. Therefore, it is not open to doubt that, for verification of the correctness of the starting assumption of the theory, there is required a mathematical apparatus which will make it possible to obtain results undistorted by any kind of supplementary simplifications.

A new method has recently [1, 3] been proposed for the numerical solution of the equations of turbulent boundary layers on solid surfaces. This method is based on the concept of a starting system of transfer equations in new independent variables (analogous to a Mises transform), and on the use of a finite-difference scheme which, thanks to selected variables, assures a uniform accuracy of the solution over the whole length of the boundary layer. The method [1-3] is also effective in that it permits a considerable economy of machine time and facilitates substantially the analysis of the calculated results. With certain modifications, this method can be applied also to liquid–gas and liquid–liquid systems, which opens up broad prospects for further development of the theory of industrial chemical processes. On the basis of a modified method [1-3] we shall expound below a scheme for calculation of turbulent hydrodynamic and diffusion boundary layers in a stream of a gas flowing around a laminar film of liquid.

We consider a vertical plane-parallel channel with a width of 2H, over whose walls flows a thin laminar film of liquid. Let it be assumed that a flow of turbulent gas moves through the channel in the direction of motion of the film. We shall assume that, in the inlet cross section of the channel, the velocity profile of the gas is planar, with a mean velocity of U_0, and that the initial velocity profile in the film is described by the Nusselt formula. We shall also assume that the mean flow velocity of the film is small in comparison with U_0. If the wave-formation effect is neglected, then, in a given cross section x, the velocity of the liquid, as can easily be shown, will be described by the expression

$$v_{liq_x} = {}^3/_2\, Q_{liq} y\,(\beta(x) - y) \quad (0 \leqslant y \leqslant h(x)), \tag{1}$$

where x is the coordinate along the axis parallel to the walls of the channel; y is the transverse coordinate (the origin of the calculation is taken at the inlet of the channel at one of its walls); Q_{liq} is the flow rate of the liquid in the film; β (x) is a coefficient taking into account the interaction of the flows of liquid and gas (in the absence of interaction, $\beta = 2$); h(x) is the thickness of the film.

At the interface, there exists a condition of equality of the tangential stresses and velocities

$$\mu_g \partial v_{g\tau} / \partial n|_{y=h} = \mu_{liq} \partial v_{liq\tau} / \partial n|_{y=h}; \quad v_{g\tau}|_{y=h} = v_{liq\tau}|_{y=h}, \tag{2}$$

where n is the external normal to the surface of the film; μ_g and μ_{liq} are the dynamic viscosities of the gas and the liquid. Using these relationships, as well as expression (1) and the equation

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$$h^3(x) - {}^3/_2\beta(x)h^2(x) + 1 = 0,$$

flowing out of the conditions of the conservation of Q_{liq} in the film, we eliminate the unknowns h(x) and $\beta(x)$. As a result, we obtain a formula which, in the dimensionless variables

$$v = v_{g_x}/U_0; \quad x' = x/H; \quad y' = y/H$$

has the form

$$2\varepsilon_1^3\varepsilon_2^3\tau^3 - \varepsilon_1\varepsilon_2^2\tau^2v_0^2 + 18\varepsilon_1^2\varepsilon_2\tau v_0 + 27\varepsilon_1^2 - 8v_0^3 = 0, \tag{3}$$

where

$$\tau = \frac{\partial v}{\partial y'}\bigg|_{y'=h_0/H}; \quad v_0 = v\bigg|_{y'=h_0/H}; \quad \varepsilon_1 = \frac{\overline{U}_{liq}}{U_0}; \quad \varepsilon_2 = \frac{\mu_g}{\mu_{liq}}\frac{U_0}{\overline{U}_{liq}}\left(\frac{h_0}{H}\right)^2;$$
$$\overline{U}_{liq} = \frac{Q_{liq}}{h_0};$$

h_0 is the initial thickness of the film. In the derivation of formula (3) the assumption $h_0 - h \ll H$ was used.

We carry through the replacement of the independent variables

$$y' \to \omega = \psi / \psi_e; \quad x' \to \xi \equiv x',$$

where ψ is a function of the flow, equal to zero at the surface of the film, and equal to ψ_e at the external limit of the boundary layer. In the variables (ξ, ω), the equations of the boundary layer have the form

$$v\frac{\partial v}{\partial \xi} - \frac{\omega}{\psi_e}\frac{d\psi_e}{\partial \xi}v\frac{\partial v}{\partial \omega} = -\frac{\partial P}{\partial \xi} + \frac{v}{\psi_e^2}\frac{\partial}{\partial \omega}\left(\nu_{eff}\, v\frac{\partial v}{\partial \omega}\right); \quad \frac{\partial P}{\partial \omega} = 0, \tag{4}$$

where P is the dimensionless pressure (connected with the true pressure by the relationship $P = p/\rho_g U_0^2$; $\nu_{eff} = (\nu + \nu_T)/U_0H$; ν and ν_T are the molecular and turbulent kinematic viscosities; ρ_g is the density of the gas. According to the semi-empirical theory of Prandtl [4], ν_T is determined by the expression:

$$\nu_T = \varkappa y^2 \partial v_{g_x}/\partial y, \tag{5}$$

where $\varkappa = 0.4$ is an empirical constant.

At the external limit of the boundary layer, there exists the condition

$$v(\xi, \omega = 1) = U_{ext}(\xi), \tag{6}$$

where $U_{ext}(\xi)$ is the velocity of the external flow, determined from the equation of motion of an ideal liquid:

$$U_{ext}\, dU_{ext}/d\xi = \partial P / \partial \xi. \tag{7}$$

The pressure can be found from the condition of the constancy of the gas flow rate:

$$\psi_e\int_0^1 \frac{d\omega}{v} + \frac{1-\psi_e}{U_{ext}} = 1. \tag{8}$$

It is evident that formulas (6) and (7) have meaning only when $\psi_e < 1$. If $\psi_e = 1$, there is no external flow and, at the axis of the channel, the condition is fulfilled

$$\frac{\partial v}{\partial \omega}(\xi, \omega = 1) = 0. \tag{9}$$

In the determination of ψ_e there is a certain indeterminacy, since the distribution of the gas phase between the region of external flow, where the effect of viscosity can be neglected, and the region of the boundary layer, is approximate. To eliminate this indeterminacy, we assume [1]

$$v(\xi,\ 1-\Delta\omega) = 0.99U_{ext}(\xi). \tag{10}$$

In the process of numerical calculation, the quantity $\Delta\omega$ is usually selected equal to the mesh spacing in the direction ω [1]. Relationship (10) can be used to find ψ_e from the equation of motion, written in the finite-difference form.

Scheme of Numerical Solution. The transfer equations can be written as one equation for all processes of the form

$$\frac{\partial\Phi}{\partial\xi} + b\omega\frac{\partial\Phi}{\partial\omega} = f(\Phi) + \frac{\partial}{\partial\omega}\left(c\frac{\partial\Phi}{\partial\omega}\right),$$

where Φ is the substance transferred; b and c are coefficients, common to all transfer equations, and the function $f(\Phi)$ depends on the special characteristics of a concrete process. The region of change of the variables (ξ, ω) is the semi-infinite band $(0 \ll \omega \ll 1,\ 0 \ll \xi < \infty)$. To construct a mesh with the mesh points (i, k), we draw two families of parallel lines: ω_i = const (i = 1, 2,..., N + 1), ξ_k = const (k = 1, 2,...). The mesh points (i = 1, k) will be called limiting, and the mesh points (i = N + 1, k) will be called initial. Without dwelling on the details of the expression for each term of Eq. (4) in a discrete form, we give the difference equation for the velocity in the general form:

$$v_{i,k} = A_{i,k}v_{i+1,k} + B_{i,k}v_{i-1,k} + C_{i,k} + D_{i,k}(\partial P/\partial\xi)_k, \qquad i = 2, 3, \ldots, N;\quad k = 1, 2, \ldots \tag{11}$$

The coefficients $A_{i,k}$, $B_{i,k}$, $C_{i,k}$, and $D_{i,k}$ contain the unknown quantity ψ_{ek} $(d\psi e/d\xi)_k$, which can be found from Eq. (11) at i = N. Here it is necessary to use the condition

$$v_{N,k} = 0.99v_{N+1,k}, \tag{12}$$

which flows out of relationships (6) and (10)

Using the method of differences [5], we transform (11) to the form

$$v_{i,k} = S_{i,k}v_{i+1,k} + Q_{i,k} + R_{i,k}(\partial P/\partial\xi)_k,\ i = 2,3,\ldots,N;\quad k = 1, 2, \ldots, \tag{13}$$

where $S_{i,k}$, $Q_{i,k}$, and $R_{i,k}$ are determined by the recurrent formulas

$$S_{i,k} = \frac{A_{i,k}}{1 - B_{i,k}S_{i-1,k}};\quad Q_{i,k} = \frac{B_{i,k}Q_{i-1,k} + C_{i,k}}{1 - B_{i,k}S_{i-1,k}};\quad R_{i,k} = \frac{B_{i,k}R_{i-1,k} + D_{i,k}}{1 - B_{i,k}S_{i-1,k}}. \tag{14}$$

To find the coefficients (14) for all indices i = 2, 3,..., N, it is necessary to use the boundary condition (3) to determine the coefficients $S_{1,k}$, $Q_{1,k}$, $R_{1,k}$, which will be done below. Assuming that the coefficients (14) are known, we transform Eq. (13):

$$v_{i,k} = F_{i,k} + H_{i,k}(\partial P/\partial\xi)_k,\quad i = 2, 3, \ldots, N+1;\ k = 1, 2, \ldots, \tag{15}$$

where

$$F_{i,k} = S_{i,k}F_{i+1,k} + Q_{i,k},\quad H_{i,k} = S_{i,k}H_{i+1,k} + R_{i,k}. \tag{16}$$

Using expressions (7) and (8) in the discrete form

$$U_{\text{ext}_k} = U_{\text{ext}_{k-1}} - \frac{\Delta\xi}{U_{\text{ext}_{k-1}}}\left(\frac{\partial P}{\partial \xi}\right)_k, \tag{7a}$$

$$\psi_{e_k} \sum_{i=1}^{N} \frac{\Delta\omega}{v_{i,k} + v_{i+1,k}} + \frac{1-\psi_{e_k}}{U_{\text{ext}_k}} = 1, \tag{8a}$$

and Eq. (15), it is possible to express $(\partial P/\partial \xi)_k$ as a function of the coefficients $F_{i,k}$ and $H_{i,k}$. We shall not give here the details of this procedure and the explicit form of the function $(\partial P/\partial \xi)_k$.

Thus, the problem of finding the distribution of the velocities and the pressures reduces to determination of the coefficients $F_{i,k}$ and $H_{i,k}$. For this, as follows from the recurrent relationships (16), it is required to know $F_{N+1,k}$

$$F_{N+1,k} = U_{\text{ext}_{k-1}}, \quad H_{N+1,k} = -\Delta\xi/U_{\text{ext}_{k-1}}.$$

With $\psi_e = 1$, the coefficients $F_{N+1,k}$ and $H_{N+1,k}$ are found by solutions of Eqs. (9), (11), and (13). Here, Eq. (11) must be taken at $i = N+1$, and Eq. (13) at $i = N$.

In order not to introduce a too small mesh spacing (the necessity of a small mesh in the classical treatment [5] is due to the large velocity gradients near the interface) and, simultaneously, to take account of the true distribution of the velocities at the surface of the film, we use the method of so-called fictitious velocities, proposed for the case of a solid wall in [1-3]. The essence of the method consists in the fact that the analytical solutions for v and $\partial v/\partial\omega$, obtained for the region near the interface under the assumption that the inertial terms within the limits of this region do not play a substantial role, are approximated at the point $\omega = \Delta\omega/2$ by the difference relationships:

$$(v_{1,k} + v_{2,k})/2 = v_{3/2,k}, \quad (v_{2,k} - v_{1,k})/\Delta\omega = (\partial v/\partial\omega)_{3/2,k}, \tag{17}$$

where the index $^3/_2$ corresponds to $\omega = \Delta\omega/2$. Here it must be kept in mind that $v_{1,k}$ is not the true velocity at the interface, but some fictitious velocity, introduced to make it possible to apply the principle of a linear approximation right up to the surface of the film.

If the above analytical solutions for v, $\partial v/\partial\omega$ are linearized with respect to τ and v_0, using Eq. (3) (also linearized with respect to τ and v_0) to eliminate v_0 and τ from expressions (17), there is obtained a relationship connecting the quantities $v_{1,k}$ and $v_{2,k}$ in a linear fashion

$$v_{1,k} = S_{1,k} v_{2,k} + Q_{1,k}. \tag{18}$$

In this expression, the coefficients $S_{1,k}$ and $Q_{1,k}$ are true functions of the quantities $v_{1,k-1}$, $v_{2,k-1}$, $v_{0,k-1}$ and τ_{k-1}, determined for the $(k-1)$-th mesh. The explicit form of these coefficients is very cumbersome, and we shall not give it here. From the operations decribed above it follows also that $R_{1,k} = 0$.

At the present time, following the above scheme, an electronic computer is being used to calculate the distributions of the velocities and pressures in a turbulent gas flowing around a laminar film, over a broad interval of values of the parameters ε_1 and ε_2 and of the Reynolds number in the gas. The results of the calculations and a comparison of them with experimental data will be presented in a separate communication.

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