THE USE OF HARMONIC SIGNALS FOR STUDYING LONGITUDINAL HYDRODYNAMIC MIXING IN POROUS MEDIA WITH STAGNANT ZONES

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Hydrodynamic mixing in porous media may be studied not only by sudden (delta-shaped) injection of a marker into the flow [7], but also with the aid of harmonic signal.† When the latter method is applied the effective diffusion coefficient may be determined in two ways: from the signal amplitude at the exit (D_a) and from the phase shift (D_f). Experimental determination of the dispersion (diffusion) coefficient in these two ways [2] yielded different results, namely, D_f exceeded D_a by 30%.

The dispersion coefficient is usually found by identifying the results derived from a diffusion model with those obtained from a model that describes a definite mixing mechanism, for example, from the model of a series of ideally mixed cells. We shall therefore first consider a homogeneous flow of velocity u to which an impurity or a marker with diffusion coefficient D has been added. In this case the distribution of the marker concentration is described by the usual one-dimensional diffusion equation for convective transfer. Let the marker be added in the plane x = 0 at a rate $Ae^{i\omega t}$ that varies harmonically with time.

The marker concentration in an arbitrary point at an arbitrary moment of time equals

$$c(x, t) = A \int_{-\infty}^{t} e^{i\omega\tau} G(x, t-\tau)\, d\tau, \tag{1}$$

$$G(x, t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left[-\frac{(x-ut)^2}{4Dt}\right]. \tag{2}$$

To analyze the change in amplitude and phase shift, it is indispensable to find the ratio c(L,t)/c(0,t), where L denotes the distance between the point where the marker is added and the point where the measurement is done. Calculation of integral (1), on the assumption that $4D\omega/u^2 \ll 1$ yields the following relationship for the ratio sought‡:

$$\frac{c(L, t)}{c(0, t)} = \exp\left[-i\frac{L\omega}{u}\left(1-\frac{2D^2\omega^2}{u^4}\right) - \frac{LD\omega^2}{u^3}\right]. \tag{3}$$

We shall now consider a chain of ideally mixed cells with stagnant zones as described in [1]. Let the concentration in the flow zone of cell zero vary harmonically with time, i. e., equal exp ($i\omega t$). The marker concentration $c_{1k}(t)$ in the flow zone of the kth cell and the average concentration $c_{2k}(t)$ in the stagnant zone of the same cell are determined by the set of equations derived from the mass balance:

$$\begin{gathered} dc_{1k}/dt + (\lambda+\nu)c_{1k} - \nu c_{2k} - \lambda c_{1,k-1} = 0, \\ dc_{2k}/dt + \gamma c_{2k} - \gamma c_{1k} = 0. \end{gathered} \tag{4}$$

The meaning of parameters λ, ν, γ is the same indicated in paper [1]:

$$\lambda = \frac{q}{(1-\alpha)V}; \quad \nu = \frac{p}{(1-\alpha)V}; \quad \gamma = \frac{p}{\alpha V}. \tag{5}$$

The equation given above for the concentration in the flow zone of cell zero is a boundary condition for the set of equations. No initial conditions are needed because we shall consider the stationary process.

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†The concentration of the marker at the inlet is varied sinusoidally.

‡Here, the medium is taken to be infinite. Analysis shows that the results for a semiinfinite medium hardly differ from (3), if L is sufficiently large.

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Eliminating the concentrations in the stagnant zones from the set (4), we derive the following system of recurrent equations:

$$d^2c_{1k} / dt^2 + (\lambda + \nu + \gamma) dc_{1k} / dt + \lambda\gamma c_{1k} \tag{6}$$
$$-\lambda\, dc_{1,\,k-1} / dt - \lambda\gamma c_{1,\,k-1} = 0.$$

The concentrations in an arbitrary cell and in cell zero vary with time according to the same law, for, it follows from (6) that

$$c_{1k}(t) = \frac{\lambda(i\omega + \gamma)}{-\omega^2 + i\omega(\lambda + \nu + \gamma) + \lambda\gamma} c_{1,\,k\text{-}1}(t). \tag{7}$$

Repeating the operation indicated in (7), we find the ratio between the concentration in the last (n^{th}) cell and the concentration at the chain inlet

$$\frac{c_{1n}(t)}{c_{10}(t)} = \left[\frac{\lambda(i\omega + \gamma)}{-\omega^2 + i\omega(\lambda + \nu + \gamma) + \lambda\gamma}\right]^n. \tag{8}$$

We shall consider two special cases.

I. Let $\omega \ll \lambda, \nu, \gamma$. After several simplifications we get:

$$\ln\frac{c_{1n}}{c_{10}} \approx -n\omega^2\left[\frac{V^2}{2q^2} + \frac{\alpha^2V^2}{pq}\right] - \frac{in\omega V}{q}\left[1 - \omega^2\left(\frac{V^2}{3q^2} + \frac{\alpha^2V^2}{qp} + \frac{\alpha^3V^2}{p^2}\right)\right]. \tag{9}$$

From a comparison of the phase shift calculated for the diffusion model (formula (3)) with the phase shift calculated for the model of cells with stagnant zones (formula (9)) it follows, first of all, that

$$V / q = l / u, \tag{10}$$

where l = L/n denotes the length of a cell, and u the mean flow rate. Substituting this into (9) and comparing again with (3), we find the dispersion coefficient determined from the change in amplitude

$$D_a = ul / 2 + ul\alpha^2 q / p. \tag{11}$$

This expression is identical to that derived in paper [1] for the dispersion coefficient measured with a delta-shaped pulse.

Comparison of the phase shifts as predicted by formulas (3) and 9)yields the following expression for the dispersion coefficient:

$$D_f = ul\sqrt{{}^1/_6 + \alpha^2 q / 2p + \alpha^3 q^2 / 2p^2}. \tag{12}$$

Consequently, amplitude and phase analyses yield different expressions. We shall investigate the relationship between D_a and D_f. We shall consider the case that the stagnant zones play the most important role, i. e., the intensity p of the exchange between the flow and stagnant zones is very low. Then, only the last few terms in formulas (11) and (12) are important, and the ratio of the two dispersion coefficients equals

$$D_a / D_f = \sqrt{2\alpha}. \tag{13}$$

If the stagnant zone occupies less than half the cell, it is evidently more probable that this ratio is lower than unity, and, consequently, $D_a < D_f$. Precisely this relationship was found experimentally by Kramers and Alberda [2].

It is interesting to find out how the ratio between D_a and D_f varies when the role played by stagnant zones is reduced. This can be found by extrapolating (11) and (12) to the limit $\alpha \to 0$. This limiting case evidently corresponds to complete absence of stagnant zones. Then

$$D_a / D_f = \sqrt{{}^3/_2} \tag{14}$$

and, consequently the relationship between D_a and D_f is reversed:

$$D_a > D_f$$

Formula (14) is valid for ideally mixed cells without stagnant zones. It turns out that even in this simplest model, amplitude and phase analysis yield different values for the effective diffusion coefficients. This discrepancy between D_a and D_f can be easily explained, if we pass from the difference equation

$$\frac{l}{u}\frac{dc_k}{dt} = c_{k-1} - c_k, \tag{15}$$

which corresponds to the model of ideally mixed cells without zones, to the limiting case of a continuous diffusion model. Restricting ourselves to the first three terms in expansion of the difference $(c_{k-1} - c_k)$, we get

$$\frac{\partial c}{\partial t} = -u\frac{\partial c}{\partial x} + \frac{ul}{2}\frac{\partial^2 c}{\partial x^2} - \frac{ul^2}{6}\frac{\partial^3 c}{\partial x^3}. \tag{16}$$

If the signal frequency is sufficiently low, and only the most important terms are retained, Eq. (16) may be converted into

$$\partial c / \partial t = -u\, \partial c / \partial x + D\, \partial^2 c / \partial x^2, \tag{17}$$

where $D = ul/2$. In this way we get the well-known equation for the coefficient of longitudinal diffusion.

From (3) it follows that the discrepancy between D_a and D_f is related to a small term of a higher order in the frequency, so that it is inevitable to utilize Eq. (16). We search for a solution of the form

$$c(x, t) = a(x)e^{i\omega t}. \tag{18}$$

Solution of the characteristic equation

$$Ek^3 - Dk^2 + uk + i\omega = 0, \quad E = ul^2/6 \tag{19}$$

yields

$$k = -\omega\frac{i}{u} + \omega^2(-D/u^2) + \omega^3(i2D^2/u^4 - iE/u^3). \tag{20}$$

Substituting the values of D and E into the coefficient of ω^3, which yields the phase shift sought for, we find that this coefficient equals $il^2/3u^2$, i. e., we find agreement with the result derived from the cell model. This derivation proves that phase analysis with the cell model inevitably brings us outside of the diffusion approximation.

All these findings may be checked experimentally in the following way. The volume of the stagnant zones should evidently decrease with increasing flow rate, and the role played by stagnant regions should be reduced correspondingly. Consequently, upon a rise in flow rate the inequality obeyed by the two diffusion coefficients will be reversed.

II. If $\nu, \gamma \ll \omega \ll \lambda$, then

$$D_a = \frac{ul}{1-\alpha}\left[\frac{1}{2} + \frac{u^2}{(1-\alpha)^2 l^2\omega^2}\frac{p}{q}\right]; \quad D_f = \frac{ul}{(1-\alpha)\sqrt{6}}. \tag{21}$$

The two coefficients are of the same order of magnitude, since $u^2p/(1-\alpha)^2 l^2\omega^2 q \sim 1$; however, they are not equal, while $D_a > D_f$. The two coefficients are smaller than the corresponding values for the first case. Consequently, the dispersion coefficients drop with increasing signal frequency.

LITERATURE CITED

1. V. G. Levich, V. S. Markin and Yu. A. Chizmadzhev, DAN, 166, No. 6 (1966).
2. H. Kramers and G. Alberda, Chem. Eng. Sci., 2, 173 (1953).