# HYDRODYNAMIC MIXING IN A POROUS MEDIUM MODEL WITH STAGNANT ZONES

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Longitudinal mixing during the flow of liquids and gases through porous media is most often described with the model of a chain of ideal mixer cells [1]. This model is applicable at sufficiently high velocities, where the flow is turbulent. However, several experimental findings are not conclusively explained by this model of ideal mixer cells [2]. It was supposed that mixing does not take place throughout the entire cell volume but solely in part of this volume. The other part constitutes a stagnant zone. By stagnant zone is meant that part of the porous space in which exchange with the liquid in the flow zone is hampered. It is obvious that this model correctly represents some features of liquid flow through porous media. However, when the author of paper [2] established that a chain of ideal mixer cells with stagnant zones is equivalent to the classical diffusion model, he prematurely changed over from the discrete to the continuous description of a porous medium. As a result, the contribution from fixing in the flow zone of the cells to the effective coefficient of longitudinal got lost.

The aim of the present study is to give a more correct discussion of the model of a chain of ideal mixer cells with stagnant zones, and to establish the relation with the usual diffusion model.

Let the porous medium consist of a chain of identical cells of volume V and length $l$. Part of this volume, $\alpha V$, is occupied by a stagnant zone, and the rest, $(1-\alpha)V$, by the flow zone in which ideal mixing takes place. The two zones exchange material—liquid or gas—at the rate p.[†] The space velocity in the flow zones equals q. The time in which an arbitrary liquid particle traverses the entire chain of cells is a stoichastic variable depending on the time needed to traverse a single cell. It is therefore indispensable to study first the mechanism of liquid flow in a single cell.

We shall consider the cell whose characteristics are given above. The time t in which a liquid (or gas) particle traverses the cell- the so-called age of the particles at the exit of the cell [3]—is a stoichastic variable. We shall search for the distribution of this variable. We suppose that at the moment $t = 0$ a unit amount of a neutral admixture (marker) is added at once ($\delta$-function) to the flow entering the cell.

The concentration of the admixed substance in the flow zone will be denoted by $c_1(t)$, that in the stagnant zone by $c_2(t)$. The density of the age distribution of the particles $f(t)$ near the exit can be simply expressed by $c_1(t)$. According to the definition of ideal mixing [3], the amount of the admixture carried away from the cell per unit time equals $qc_1(t)$. The age of the admixture carried away evidently equals t. Since the total amount admixed equals unity, it follows that

$$f(t) = qc_1(t). \tag{1}$$

Equations for the determination of $c_1(t)$ and $c_2(t)$ are obtained by drawing up the mass balances:

$$\begin{gathered} dc_1 / dt + (\lambda + \nu) c_1 - \nu c_2 = 0, \\ dc_2 / dt + \gamma c_2 - \gamma c_1 = 0. \end{gathered} \tag{2}$$

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[†] The exchange mechanism may be either convective or diffusive. It is not specified in this paper.

Here, we employ the notation:

$$\lambda = q / (1 - \alpha) V, \quad \nu = p / (1 - \alpha) V, \quad \gamma = p / \alpha V. \tag{3}$$

The boundary conditions for the set of equations (2) are

$$c_1(0) = 1 / (1 - \alpha) V, \quad c_2(0) = 0. \tag{4}$$

It is indispensable to find the average $\bar{t}$ and the variancy $\sigma^2$ of the stoichastic variable t—the particle age at the cell exit. This is done with equations (1)-(4), which yield

$$\bar{t} = V / q, \quad \sigma^2 = V^2 / q^2 + 2\alpha^2 V^2 / pq. \tag{5}$$

The value thus found for the average residence time in the cell considered exactly coincides with the average residence time in an ideal mixer cell without stagnant zone [3]. Consequently, the average time in which an arbitrary particle traverses the cell does not depend on the relative volume of the stagnant zone nor on the intensity of the exchange between the flow and the stagnant zones.

The expression derived above cannot be extended to p = 0. Solution for p = 0 yields

$$\bar{t} = (1 - \alpha) V / q, \quad \sigma^2 = (1 - \alpha)^2 V^2 / q^2. \tag{6}$$

We shall now consider a chain consisting of n identical cells. The time T needed to traverse the entire chain is the sum of the residence times in the various cells $t_k$:

$$T = \sum_{k=1}^{n} t_k. \tag{7}$$

The stoichastic variables are independent, have identical distributions and finite nonzero variancies. Consequently,

$$\bar{T} = n\bar{t} = n\nu / q, \quad \overline{(T - \bar{T})^2} = n\sigma^2 = nV^2 / q^2 + 2\alpha^2 nV^2 / pq, \tag{8}$$

and from the central limit theory of probability theory it can be proved that if n is sufficiently large, the distribution of the stoichastic variable tends to the normal distribution:

$$h(T) = \frac{1}{\sqrt{2\pi n\sigma^2}} \exp\left[-\frac{(T - n\bar{t})^2}{2n\sigma^2}\right]. \tag{9}$$

Function h(T) denotes of the amount of admixture carried away per unit time from the last cell (chain end) at the moment T, if the amount added at the beginning of the chain at the moment T = 0 equals unity. Hence it follows that the concentration in the flow leaving the last cell equals

$$c_1(n, T) = \frac{h(T)}{q} = \frac{1}{q\sqrt{2\pi n\sigma^2}} \exp\left[-\frac{(T - n\bar{t})^2}{2n\sigma^2}\right]. \tag{10}$$

Hydrodynamic mixing in the porous medium considered can also be described by means of a diffusion model with an effective diffusion coefficient D. Assume that the liquid flows through the porous medium with velocity u. By velocity is meant the true mean flow velocity found by averaging over the entire ensemble of liquid particles present in the porous medium, and not the filtration rate calculated as the ratio of the volume passing per unit time to the cross-sectional area of the porous medium. If an amount of admixture equal to A is added at once to the liquid entering the porous medium at T = 0, then, the concentration in a point a distance L away from the inlet as a function of time will be given by the formula

$$c(L, T) = \frac{A}{\sqrt{4\pi DT}} \exp\left[-\frac{(L - uT)^2}{4DT}\right]. \tag{11}$$

If the distance L is sufficiently large, i.e. if L >> D/u, the concentration as a function of time is determined mainly by the numerator of the exponent. In all other cases, T may be replaced by L/u without this causing any substantial error. As a result, formula (11) transforms into

$$c(L, T) = \frac{A}{\sqrt{4\pi DL/u}} \exp\left[-\frac{(T - L/u)^2}{4DL/u^3}\right]. \tag{12}$$

Hence it follows that the model of ideal mixer cells with stagnant zones and the diffusion model yield qualitatively identical results for sufficiently large distances. To achieve quantitative agreement it is necessary to relate the parameters of the two models with the relationships

$$q^2 n\sigma^2 = 2DL / A^2 u, \quad n\bar{t} = L / u, \quad n\sigma^2 = 2DL / u^3. \tag{13}$$

Hence it follows that the effective diffusion coefficient equals

$$D = ul / 2 + ul\alpha^2 q / p, \tag{14}$$

where $l = L/n$ denotes the length of a single cell, and $u = ql/V$ the mean flow rate. Only the last term was obtained in paper [2]. As mentioned above, this is due to the circumstance that mixing in the flow part of the cell is ignored. If there are no stagnant zones in the system, in other words if $\alpha = 0$, the diffusion coefficient equals $ul/2$, as is well known. Formula (14) makes allowance both for the variancy related to mixing in the flow zone and for exchange of particles between the flow and stagnant zones. It is interesting to note that the first term of (14) is identical to the diffusion coefficient, if the entire cell is occupied by the flow zone. In reality, part of the cell is occupied by the stagnant zone. We should therefore not consider formula (14) as the sum of two terms, each of which is produced by a definite mixing mechanism.

The second term of formula (14) may render the diffusion coefficient very large, and this in its turn may impose very severe restrictions on the length L needed for establishment of a normal distribution. As a result, the length of the porous medium chosen for the experimental determination of the dispersion coefficient may be too short, so that application of the diffusion model yields incorrect results. In practice, this manifests itself in that the concentration at the exit as a function of time has a very long and constant tail, and approximation of the experimental dependence by an appropriate normal distribution yields a much too low value for the dispersion coefficient D. This was evidently the case in many experiments [4].

We shall now discuss the case of a slight exchange intensity between the flow and stagnant zones in a porous medium of short length. Low exchange intensity means that the ratios $\gamma/\lambda$, and $\gamma/\lambda \ll 1$; a short length of the porous medium expressed in terms of the cell chain model means that $1 \leq n \leq \lambda/\gamma$. The distribution of the residence times can be conveniently calculated from the characteristic functions [5]. The characteristic function of the stoichastic variable t, which is accurate as to the main terms of the small parameters is given by the formula

$$\varphi(r) = \left(1 - \frac{\nu}{\lambda}\right)\frac{i\lambda}{r + i\lambda} + \frac{\nu}{\lambda}\,\frac{i\gamma}{r + i\gamma}. \tag{15}$$

As follows from Eq. (7), the characteristic function of the stoichastic variable T equals $\psi(r) = \varphi^n(r)$. Applying the Fourier transformation to $\varphi(r)$, we find the following expression, which is accurate as to the main terms of the distribution density of the stoichastic variable T:

$$h(T) = \left(1 - \frac{n\nu}{\lambda}\right)\frac{\lambda}{\sqrt{2\pi n}} \exp\left[-\frac{(\lambda T - n)^2}{2n}\right] + \frac{n\nu}{\lambda}\gamma \exp(-\gamma T). \tag{16}$$

The second term in this formula holds only in the range where $T > 1/\lambda$. Since precisely this range is the most interesting, we do not write down the other formulas.

The two terms in (16) may be ascribed to two mechanisms of particle passage through the system. The first mechanism results in a normal distribution of the exit concentration with an average age $n/\lambda$ and a variancy $n/\lambda^2$. The fraction of particles chosing this mechanism equals $(1 - n\,\gamma/\lambda)$. The other particles choose the second mechanism, which results in an exit concentration which drops very slowly with time. Precisely this mechanism is responsible for the "tail" of the distribution found experimentally. Consequently, hydrodynamic mixing in a not very long, porous medium cannot be described by a single diffusion model. If the experimental curve is, nevertheless, approximated by a normal distribution, it is clear that this approximation yields only the first term of the distribution (16). In this approximation the mean residence time of the particle in the porous medium equals $(1 - \alpha)$ L/u. In other words, the mean flow rate is not found equal to u, but to the higher value $u/(1-\alpha)$. The latter value implies that the particles never enter the stagnant zones. This result is not surprising at all, since the normal part of the distribution (16) describes precisely those particles that traverse the medium without staying in any stagnant zone. From comparison with (12) it follows that the effective diffusion or variancy coefficient in this approximation is found to be $D = ul/2(1-\alpha)$.

The following three parameters play a part in the problem considered: $l$, $\alpha$, and p/q. It is evident that these parameters may be determined easily from experiments. As one of the relationships relating the problem parameters to experimentally measurable parameters may serve the above derived expression for D. The position of the maximum of the distribution curve (16) yields the value of the product $(1-\alpha)L/u$. Finally, measurements of the rate at which the concentration drops in the tail enable us to determine the exponent in (16), i.e., determine the combination $\frac{p}{q}\ \frac{u}{\alpha l}$. These three equations suffice for determining the problem parameters. Several other relationship, in addition to those mentioned, can be easily derived. For example, the maximum concentration at the exit may be measured as a function of the length of the porous medium. Very interesting is the question as to whether the p/q ratio depends on the velocity. A series of experiments with different flow rates may answer this question.

## LITERATURE CITED

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