THEORY OF EXTRACTION FROM A FALLING DROP

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A discussion was given in [1–3] of the problem of extraction of material from a spherical drop of quite small radius, falling in a liquid medium under the force of gravity. It was assumed here that the extraction rate is limited by convective diffusion through the external medium, while the Péclet number corresponding to the external medium is quite large, with the result that the greatest change in concentration occurs in the diffusion boundary layer located near the surface of the drop. Kronig and Birink [4] have made a study of the effect of convective transferrence on the rate of depletion of a solution inside a falling drop for the condition that the slow stage of the process is diffusion through the drop. To find the maximum effect of the circulating motion of the liquid inside the drop, the authors simplify the equation of mass transfer, assuming that the concentration of the material dissolved in the drop remains constant along every hydrodynamic current line. It was shown by calculations made by the authors themselves that this assumption is obviously not satisfied near the surface of the drop, so that the results obtained in [4] do not apply in the case of quite large values of the Péclet number, where the greatest resistance to mass transfer is concentrated in the diffusion boundary layer. A discussion will be given in the present paper of the problem of convective diffusion of material through a drop at high values of the Péclet number, under conditions where the resistances of the continuous and the disperse phase are commensurate.

Consider a spherical drop of quite small radius R, containing dissolved material, and moving at the constant velocity U in a liquid medium. If the Péclet numbers $\mathrm{Pe}_1 = UR / D_1$ and $\mathrm{Pe}_2 = UR / D_2$ (D_1 and D_2 are the diffusion coefficients of the material inside and outside the drop respectively) are quite large, the greatest resistance to transfer of material from the drop will be concentrated in the diffusion boundary layer, located on both sides of the surface of the drop. It may be shown that for fixed values of the concentration of material on the boundaries of the diffusion layer, the relaxation time of the layer, i.e., the time required to set up a steady state distribution of concentration in the diffusion layer is $\tau_r = R(1 + \mu^*) / (2U)$ ($\mu^* = \mu_1/\mu_2$ is the ratio of the dynamic viscosities of the disperse and continuous liquid media). It was shown in [4] that the time τ_d required for the total amount of material dissolved in the drop to decrease by a factor of e has, as a lower limit, the value $0.022\ R^2/D_1$. Accordingly, we have the relation $(\tau_r/\tau_d) < 23(1 + \mu^*) / \mathrm{Pe}_1$. This means that for quite large values of the number Pe_1, there is a fairly wide range of times t which satisfy the conditions $\tau_r \ll t \ll \tau_d$. In this range of times, transfer of material in the diffusion boundary layer may be described by the steady state equation of convective diffusion for fixed values of the concentration outside the boundary layer, which are the same as the initial values of the concentration inside and outside the drop. In this case, the problem reduces to finding the solution of the system of equations:

$$\frac{v_r^{(i)}}{U}\frac{\partial c_i}{\partial r} + \frac{v_\vartheta^{(i)}}{Ur}\frac{\partial c_i}{\partial \vartheta} = \frac{1}{\mathrm{Pe}_i}\Delta c_i \qquad (i = 1, 2), \tag{1}$$

where the velocity distributions $v_r^{(i)}(r, \vartheta)$ and $v_\vartheta^{(i)}(r, \vartheta)$ are determined by the familiar formulas of Hadamard [5], and Rybczynski [6].

It has already been pointed out above that in the time intervals under discussion $t \ll \tau_d$, the concentration of material in the body of the drop (outside the diffusion boundary layer) may be assumed constant and equal to the initial value c_1^*, i.e., $c_1(r \ll 1, \vartheta) = c_1^*$. We shall also assume a fixed concentration of material in the continuous medium at a sufficient distance from the drop: $c_2(r \gg 1, \vartheta) = c_2^*$. At the surface of the drop, there will be continuous currents of material: $D_1(\partial c_2/\partial r)(1, \vartheta) = D_2(\partial c_2/\partial r)(1, \vartheta)$. It can be assumed that because of the large physical

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dissolving rates at the interface, a condition of interphase equilibrium is established instantaneously. For quite small concentrations of the material being extracted, this condition is expressed by Henry's law $c_1(1, \vartheta) = \alpha c_2(1, \vartheta)$, where α is a coefficient that depends on the temperature and the pressure.

To solve the problem posed, use was made of the Poincaré–Lighthill–Kuo method [7–11]. For the expansion parameters corresponding to the internal (1) and the external (2) phases, the values of $f_i = \sqrt{(1+\mu^*)/\mathrm{Pe}_i}$ were taken. With an accuracy up to terms of first order in f_i, the expansions are of the form:

$$c_i(\xi, \eta) = c_i^{(0)}(\xi, \eta) + f_i c_i^{(1)}(\xi, \eta) + \ldots;$$
$$r = 1 + f_i\xi + \ldots; \quad \cos\vartheta = \eta + f_i g_i(\xi, \eta) + \ldots \tag{2}$$

The functions $g_i(\xi, \eta)$ were chosen in such a way that the solutions obtained in the first approximation in the parameters f_i did not contain any singularities which could lead to divergences in the corresponding integrated values. Investigation shows that the functions $g_i(\xi, \eta)$ are to be chosen in the following form:

$$g_i(\xi, \eta) = \frac{\xi\eta}{2}\frac{1-\eta^2}{1+\eta^2}k_i, \tag{3}$$

where $k_1 = 5$, $k_2 = 3\mu^* + 2$. Substituting the expansion (2) in the initial equations and the boundary conditions, and passing to the new independent variables $\psi = \xi(1-\eta^2)$, $\tau = 2(2-\eta)^2/3$, we obtain the systems:

$$\frac{\partial c_i^{(0)}}{\partial\tau} - \frac{\partial^2 c_i^{(0)}}{\partial\psi^2} = 0; \tag{4}$$

$$\frac{\partial c_i^{(1)}}{\partial\tau} - \frac{\partial^2 c_i^{(1)}}{\partial\psi^2} = \frac{2A_i}{\sqrt{\pi\tau}}\frac{e^{-\psi^2/4\tau}}{1-\eta^2}\left\{1 + \frac{3\psi^2}{4\tau}h_i - \frac{\psi^2 k_i}{4(1+\eta^2)}\left[\frac{3\eta^2}{\tau} + \frac{\eta}{1+\eta^2} + \frac{(1-\eta^2)^2}{2\tau}\left(\frac{1}{1+\eta^2} + \frac{2\eta}{\tau}\right)\right] + \eta\frac{k_i}{1+\eta^2}\left[\eta + \frac{(1-\eta^2)^2}{2\tau}\right]\right\}, \tag{5}$$

where:

$$A_1 = \sqrt{\frac{D_2}{D_1}}(c_1^* - \alpha c_2^*)\Big/\left(\alpha + \sqrt{\frac{D_2}{D_1}}\right), \quad A_2 = -(c_1^* - \alpha c_2^*)\Big/\left(\alpha + \sqrt{\frac{D_2}{D_1}}\right),$$

and $h_1 = 1$, $h_2 = \mu^*$, with the boundary conditions:

$$c_1^{(0)}(-\infty, \tau) = c_1^*; \quad c_2^{(0)}(\infty, \tau) = c_2^*; \quad c_1^{(0)}(\psi, 0) = c_2^{(0)}(\psi, 0) = 0;$$
$$c_1^{(0)}(0, \tau) = \alpha c_2^{(0)}(0, \tau); \quad \frac{\partial c_1^{(0)}}{\partial\psi}(0, \tau) = \sqrt{\frac{D_2}{D_1}}\frac{\partial c_2^{(0)}}{\partial\psi}(0, \tau); \tag{6}$$
$$c_1^{(1)}(-\infty, \tau) = c_2^{(1)}(\infty, \tau) = 0; \quad c_1^{(1)}(\psi, 0) = c_2^{(1)}(\psi, 0) = 0;$$
$$c_1^{(1)}(0, \tau) = \alpha\sqrt{\frac{D_2}{D_1}}c_2^{(1)}(0, \tau); \quad \frac{\partial c_1^{(1)}}{\partial\psi}(0, \tau) = \frac{D_2}{D_1}\frac{\partial c_2^{(1)}}{\partial\psi}(0, \tau). \tag{7}$$

Using the solutions of these equations, it is possible to find an expression for the current of material to the drop:

$$I = 2\pi R^2\int_{-1}^{1}\left[-\frac{D_i}{R}\frac{\partial c_i}{\partial r}(1, \nu)\right]d\nu.$$

As a result of fairly unwieldy calculations, which we shall not give here, we obtain the expression:

$$I = \frac{2RD_2(c_1^* - \alpha c_2^*)}{\alpha + \sqrt{D_2/D_1}}\left\{\sqrt{\frac{8\pi\,\mathrm{Pe}_2}{3(1+\mu^*)}} - \frac{5{,}5 - \alpha(3{,}8 + 1{,}7\mu^*)}{\alpha + \sqrt{D_2/D_1}}\right\}. \tag{8}$$

The original assumption as to the smallness of the quantities $f_i = \sqrt{(1+\mu^*)/\mathrm{Pe}_i}$ does not make it possible to use Eq. (8) for a wide range of variation of the parameters occurring in it. Note, however, that in the limiting case $D_1 \to \infty$, where the mass transfer rate will be limited by the external medium, it is possible, by considering time intervals small in comparison with the time required to deplete the drop, to assume that the concentration of material

on the surface of the drop is a fixed quantity (c_1^*). In this case, the expression for the current takes the form:

$$I = 2RD_2(c_1^* - c_2^*)\left\{\sqrt{\frac{8\pi\,Pe_2}{3(1+\mu^*)}} - 1{,}7(1-\mu^*)\right\}. \tag{9}$$

Thus, Eq. (9) represents the Levich formula [1] including the next nondisappearing term in the expansion of the concentration c_2 in powers of the parameter f_2. It should however, be emphasized that the greatest advantage of Eq. (8) over the Levich formula is not simply that it applies to a wider range of values of the Péclet number, but principally that it takes account in explicit form (through α) of the relation between the current I and the physical properties of the contacting phases, and that no limitations are placed on the absolute value of the coefficient α occurring in the formula.

The solutions obtained may also be used to calculate the partial mass transfer coefficients corresponding to each of the phases.

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All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.
